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THE EVEN C^* CLIFFORD ALGEBRA

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ABSTRACT. We offer a new perspective on the theorem of Størmer demonstrating that the even and full CAR algebras are isomorphic.

Størmer [7] showed that the even CAR algebra over a separable infinite-dimensional complex Hilbert space is isomorphic to its full CAR algebra; in so doing, he neatly explained a result of Doplicher and Powers [3] to the effect that the even CAR algebra is simple. His proof involved constructing matrix units so as to realize the even CAR algebra as a UHF algebra of the same type as (and hence isomorphic to) the full CAR algebra. In this note, we present a new and rather different proof of the Størmer theorem—simpler than the original, in that it entirely circumvents the use of matrix units, and at the same time more flexible, in that it does not require separability of the underlying Hilbert space.

Actually, rather than work with CAR algebras (which rest on a choice of polarization) we prefer to work with C^* Clifford algebras (which do not). Thus, let V be an infinite-dimensional real inner product space, which we shall suppose to be complete. The C^* Clifford algebra of V is the C^* completion $C[V]$ of its complex Clifford algebra $C(V)$. The C^* algebra $C[V]$ has a unique automorphism γ restricting to V as minus the identity: the even C^* Clifford algebra $C^0[V]$ is the fixed algebra of γ, and the complementary subspace on which γ acts as minus the identity is denoted $C^1[V]$. For details on Clifford algebras refer to [2]; for C^* Clifford algebras and CAR algebras, see [1, 6].

We should perhaps point out the relationship between the C^* Clifford algebra picture and the CAR algebra picture. Let J be an orthogonal complex structure on V whose complex-linear extension to $V^{\mathbb{C}}$ has the polarization F as its i-eigenspace. The C^* Clifford algebra $C[V]$ is then naturally the CAR algebra of both V and F when each is viewed as a complex Hilbert space.

Theorem. *If the real Hilbert space V is infinite dimensional then the C^* algebras $C[V]$ and $C^0[V]$ are isomorphic.*

Proof. Let $u \in V$ be a unit vector with orthocomplement $W = u^\perp$. For plain Clifford algebras we have the decomposition

$$C^0(V) = C^0(W) \oplus u \cdot C^1(W),$$

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in which $C^0(W)$ and $u \cdot C^1(W)$ are the eigenspaces of the involution Ad_u with eigenvalues $+1$ and -1 respectively. Completion yields the decomposition

$$C^0[V] = C^0[W] \oplus u \cdot C^1[W]$$

with analogous eigenspace identifications. The homomorphism $\varphi\colon C[W] \to C^0[V]$ given by

$$\varphi(a) = \begin{cases} a, & a \in C^0[W], \\ iua, & a \in C^1[W], \end{cases}$$

is thus an isomorphism of C^* algebras— $C[W] \cong C^0[V]$. Now the infinite dimensionality of V makes itself felt; the Hilbert spaces V and W are isomorphic, whence $C[V] \cong C[W]$. This concludes our proof. □

With regard to the original proof of this theorem due to Størmer, we remark that UHF algebras themselves are classified in [5], whilst their involutive automorphisms are analyzed in [4].

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