CRINKLED FUNCTIONS AND INTERSECTIONS
WITH POLYNOMIALS

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Abstract. We prove that if \( \|f^{(n+1)}\| \) is large and \( \|f^{(n)}\|_\infty \) is bounded, then there is a polynomial \( p \) of degree \( n \) such that \( f(t) = p(t) \) has many solutions.

A classical theorem of Banach (Natanson [1]) says that if \( n_f(\alpha) = \) the number of sign changes of \( \{f(t) - \alpha\} \) on \([0,1]\) and \( f \) is a continuous function of bounded variation, then \( n_f \) is measurable and

\[
\int_{-\infty}^{\infty} n_f(\alpha) \, d\alpha = TV(f).
\]

Assume now that \( |f(t)| \leq M \) on \([0,1]\) and \( f \) is differentiable. Let \( N(f) = \text{ess sup}_{\alpha \in \mathbb{R}} n_f(\alpha). \) Since \( n_f(\alpha) = 0 \) for \( |\alpha| > M \) we obtain from (1)

\[
\int_{-M}^{M} |f'(t)| \, dt = \int_{-M}^{M} n_f(\alpha) \, d\alpha \leq 2MN(f).
\]

We may state this as an intersection theorem.

**Theorem A.** Let \( f \in C^1[0,1] \) and \( k \) be a positive integer. If \( |f(t)| \leq M \) and \( \|f''\|_1 > 2(k-1)M \) then

\[
N(f) \geq k
\]

and there is an \( \alpha \in \mathbb{R} \) so that the equation \( f(t) = \alpha \) has at least \( k \) distinct solutions on \([0,1]\). The constant \( 2(k-1)M \) cannot be replaced by any smaller number and have (3) hold.

**Proof.** Combining the hypothesis with (2) yields \( N(f) > k - 1 \). Since \( N(f) \) is an integer, (3) holds. A piecewise linear function with \( f(0) = -M \) and alternate maximum values \( M \) and minimum values \( -M \), shows that the constant cannot be improved.

This simple result quantifies the intuitive notion that a bounded crinkled function graph must agree with some horizontal line often.
Schrader in his researches on the compactness problem for $n$th order differential equations asked if there was some version of Theorem A whose hypothesis involved $f^{(n+1)}$ and whose conclusion was about intersection with polynomials of degree $n$. It is our purpose to supply such a theorem.

In some sense, we are looking for a converse of Rolle’s theorem. For example, if $f$ agrees with $p(x) = \alpha x + \beta$, a polynomial of degree 1 often, then by Rolle’s theorem $f' - \alpha$ has many zeroes. By applying Theorem A to $f'$ we would have this conclusion if $|f'| \leq M$ and $||f''||$ is large. But conversely having $||f''||$ large and $|f'| \leq M$ only gets $f' - \alpha$ with lots of zeros. Integration does not give $f(x) = \alpha x + \beta$ often. Nevertheless we are able to prove an exact replica of Theorem A in the general case.

**Theorem B.** For every nonnegative integer $n$ there is a constant $C_n > 0$ so that for $f \in AC^{(n)}(0, 1)$, the inequality

$$(4) \quad ||f^{(n+1)}||_1 > C_n \cdot (k - 1) \cdot ||f^{(n)}||_\infty$$

implies the existence of a polynomial $p$ of degree $\leq n$ so that the equation $f(t) = p(t)$ has at least $k$ distinct solutions on $[0, 1]$.

Let $f \in AC^{(n)}(0, 1)$, and put

$$(5) \quad F_n(t, u) = \frac{1}{n!} \int_u^t (t - s)^n f^{(n+1)}(s) \, ds = f(t) - \sum_{i=0}^n \frac{f^{(i)}(u)}{i!} (t - u)^i$$

for every $u, t \in [0, 1]$. Then for any fixed $u$, the equation $f_n(t, u) = \alpha$ is the equation $f(t) = p(t)$ where $p$ is a polynomial of degree $n$ in the variable $t$. If $n_{(f, n)}(\alpha)$ denotes the number of sign changes in $F_n(t, u) = \alpha$, then Theorem A reads

$$\int_{-\infty}^\infty n_{(f, n)}(\alpha) \, d\alpha = \int_0^1 \left| \frac{d}{dt} F_n(t, u) \right| \, dt = \frac{1}{(n - 1)!} \int_0^1 \left| \int_u^t (t - s)^{n-1} f^{(n+1)}(s) \, ds \right| \, dt.$$

We choose to integrate with respect to $u$ also to get

$$(6) \quad \int_0^1 \int_{-\infty}^\infty n_{(f, n)}(\alpha) \, d\alpha \, du = \frac{1}{(n - 1)!} \int_0^1 \int_u^t \left| \int_u^t (t - s)^{n-1} f^{(n+1)}(s) \, ds \right| \, dt \, du.$$

We want to relate this integral to $||f^{(n+1)}||_1$. To do this define

$$(7) \quad Tg = \int_0^1 \int_0^t \left| \int_u^t (t - s)^{n-1} g(s) \, ds \right| \, dt \, du.$$

**Proposition.** Let $B = \{g \in L^2[0, 1] \mid ||g||_2 = 1\}$ and $d_n = \inf_{g \in B} Tg$. Then there is a $g_0 \in B$ such that $d_n = Tg_0$ and $d_n > 0$.

**Proof.** Let $g_i \in B$ such that $Tg_i \to d_n$. By weak compactness of $B$ we may assume that $g_i$ converges weakly to $g_0 \in B$, i.e.,

$$\lim_{i \to \infty} \int_0^1 g_i f \, ds = \int_0^1 g_0 f \, ds$$

for all $f \in L^2[0, 1]$. Now

$$\int_u^t (t - s)^{n-1} g_i(s) \, ds = \int_0^1 (t - s)^{n-1} \chi_{[u, t]} g_i(s) \, ds$$
and so
\[
\int_{t}^{u} (t-s)^{n-1} g_i(s) \, ds \to \int_{t}^{u} (t-s)^{n-1} g_0(s) \, ds
\]
for every \( t, u \in [0, 1] \). Since \( |f'_u(t-s)^{n-1} g_i(s) \, ds| \leq \|g_i\|_1 \leq \|g_i\|_2 = 1 \), by the Lebesgue dominated convergence theorem we have \( \lim_i \, Tg_i = Tg_0 = d_n \).
We now show that \( d_n > 0 \).

If \( d_n = 0 \) then by (7)
\[
\int_{t}^{u} (t-s)^{n-1} g_0(s) \, ds = 0 \quad (t, u \in [0, 1]).
\]
Since \( g_0 \in L^2 \subset L^1 \), there is \( f \in AC^{(n)}(0, 1) \) such that \( f^{(n+1)} = g_0 \) a.e. Then
\[
f(t) - \sum_{i=0}^{n-1} \frac{f^{(i)}(u)}{i!} (t-u)^i = \frac{1}{(n-1)!} \int_{u}^{t} (t-s)^{n-1} f^{(n+1)}(s) \, ds = 0
\]
for every \( t \) and \( u \) and thus \( f \) is a polynomial of degree \( \leq n \). Therefore \( g_0 = f^{(n+1)} = 0 \) a.e., contradicting the equation \( \|g_0\|_2 = 1 \).

**Proof of Theorem B.** For \( n = 0 \) Theorem B is Theorem A so we may assume \( n \geq 1 \). Let \( C_n = 4/d_n \) and \( \|f^{(n+1)}\|_1 > C_n \cdot (k - 1) \cdot \|f^{(n)}\|_\infty \). Then by the Cauchy-Schwarz inequality
\[
\|f^{(n+1)}\|_1 \leq \|f^{(n+1)}\|_2 \quad \text{so} \quad \|f^{(n+1)}\|_2 > C_n \cdot (k - 1) \cdot \|f^{(n)}\|_\infty.
\]
Combining (8), (6), and the proposition, we obtain
\[
\int_{0}^{\infty} \int_{-\infty}^{\infty} n_{(f,n)}(\alpha) \, d\alpha \, du \geq \frac{d_n}{n!} \|f^{(n+1)}\|_2 \geq \frac{4}{n!} \|f^{(n)}\|_\infty \cdot (k - 1).
\]
Since, by (5),
\[
F_n(t, u) = \frac{f^{(n)}(\xi)}{n!} (t-u)^{n} - \frac{f^{(n)}(u)}{n!} (t-u)^{n},
\]
we have \( |F_n(t, u)| \leq \frac{2}{n!} \|f^{(n)}\|_\infty \). The left-hand side of (9) may be estimated by \( \frac{4}{n!} \|f^{(n)}\|_\infty \cdot \text{ess sup} \, n_{(f,n)}(\alpha) \). Thus \( \text{ess sup} \, n_{(f,n)} > k - 1 \), and the result follows.

Results complementary to ours are given in Agronsky et al. [2] who show that if intersections with polynomials are few, then certain derivatives must be monotone on subintervals.

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**References**


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