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SIMPLICITY OF CROSSED PRODUCTS OF C^*-ALGEBRAS

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ABSTRACT. Let (A, G, α) be a C^*-dynamical system and let G be a discrete group. When G is a cental shift in (A, G, α), we show that A is G-simple (resp. G-prime) if and only if the C^*-crossed product $A \times_\alpha G$ is simple (resp. prime).

1. INTRODUCTION

Let (A, G, α) be a C^*-dynamical system. Our aim is to continue the investigation of the relationship between the property of the C^*-dynamical system (A, G, α) and the ideal structure of the corresponding C^*-crossed product $A \times_\alpha G$. This problem first appeared in [6] and has been studied in [2, 3, 4, etc.]. Olesen and Pedersen [7] gave the necessary and sufficient condition of simplicity of C^*-crossed products by locally compact abelian groups. When G is a discrete group and A is an AF-algebra, Elliott [2] showed that if α is properly outer and A is G-simple then the reduced product $A \times_{\alpha r} G$ is simple. Later, Kawamura and Tomiyama [3] obtained the same result when A is an abelian C^*-algebra. In this paper we study simplicity of the C^*-crossed product $A \times_\alpha G$ for a general C^*-algebra when G is a discrete group.

Let (A, G, α) be a C^*-dynamical system and G be a discrete group. Let A'' be the enveloping von Neumann algebra of a C^*-algebra A. Then the action $\alpha: g \to \alpha_g$ induces the action $\alpha'': g \to \alpha''_g$ on A''. Then (A'', G, α'') becomes a W^*-dynamical system. It is said that G is a central shift in (A, G, α) if there exists a central projection p in A'' such that $\sum_{g \in G} \alpha''_g(p) = 1$ and $\alpha''_g(p)p = 0$ for every $g \neq e$ in G, where e is the identity of G (see [1]). When G is a central shift in (A, G, α), we show that A is G-simple if and only if the C^*-crossed product $A \times_\alpha G$ is simple.

2. MAIN RESULT

Let (M, G, α) be a W^*-dynamical system and $M \subset B(H)$ for a Hilbert space H. The W^*-crossed product $M \times_\alpha G$ is the von Neumann algebra on

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$L^2(G, H)$ generated by $\{\pi_\alpha(x), \lambda_g | x \in M, g \in G\}$, where

$$(\pi_\alpha(x)\xi)(s) = \alpha_{s^{-1}}(x)\xi(s), \qquad (\lambda_g\xi)(s) = \xi(g^{-1}s)$$

for every x in M, s, g in G, and ξ in $L^2(G, H)$.

Lemma 1. *Let (A, G, α) be a C^*-dynamical system. Let I be an α-invariant norm closed two-sided ideal of A. Then $I \times_\alpha G$ is a norm closed two-sided ideal of $A \times_\alpha G$.*

Proof. Let $(\pi \times \lambda)$ be the universal representation of $A \times_\alpha G$ induced by some covariant representation (π, λ, H) of the C^*-dynamical system (A, G, α), and let $(A \times_\alpha G)''$ be the enveloping von Neumann algebra of $A \times_\alpha G$. Let I be an α-invariant norm closed two-sided ideal of A. Let p be a projection in the center of $\overline{\pi(A)}^{\sigma w}$ such that

$$\overline{\pi(I)}^{\sigma w} = \overline{\pi(A)}^{\sigma w} p$$

where $\overline{(\ )}^{\sigma w}$ means the σ-weak closure of ( ). It is clear that p is contained in $(A \times_\alpha G)''$. Since I is α-invariant, we have that $\lambda_g p \lambda_g^* = p$ for every g in G. Then we put for every f in $L^1(G)$

$$\lambda_f = \int_G f(g)\lambda_g \, dg$$

where dg is the Haar measure on G. For x in A and f in $L^1(G)$ we obtain

$$\pi(x)\lambda_f p = \pi(x)p\lambda_f = p\pi(x)\lambda_f .$$

Since $(A \times_\alpha G)''$ is generated by $\{\pi(x)\lambda_f | x \in A, f \in L^1(G)\}$, p is contained in the center of $(A \times_\alpha G)''$. Also, since $(I \times_\alpha G)''$ is generated by $\{\pi(x)\lambda_f | x \in I, f \in L^1(G)\}$, we get

$$\overline{(\pi \times \lambda)(A \times_\alpha G)}^{\sigma w} p = \overline{\pi \times \lambda(I \times_\alpha G)}^{\sigma w} = (I \times_\alpha G)'' .$$

It follows that $I \times_\alpha G$ is a norm closed two-sided ideal of $A \times_\alpha G$.

Theorem 2. *Let (A, G, α) be a C^*-dynamical system and let G be a discrete group. Let G be a central shift in (A, G, α). Then A is G-simple if and only if the C^*-crossed product $A \times_\alpha G$ is simple.*

Proof. First suppose that A is G-simple. Let A'' be the enveloping von Neumann algebra of A, and let (A'', G, α'') be the W^*-dynamical system induced by the C^*-dynamical system (A, G, α). Since G is a central shift in (A, G, α), by [1] there exists an *-isomorphism ϕ from the enveloping von Neumann algebra $(A \times_\alpha G)''$ of $A \times_\alpha G$ onto the W^*-crossed product $A'' \times_{\alpha''} G$. Let J be a nonzero norm closed two-sided ideal of $A \times_\alpha G$. There exists a projection p_0 in the center of $(A \times_\alpha G)''$ such that

$$\overline{J}^{\sigma w} = (A \times_\alpha G)'' p_0$$

where $\overline{J}^{\sigma w}$ denotes the σ-weak closure of J. Then we have

$$\phi(\overline{J}^{\sigma w}) = (A'' \times_{\alpha''} G)\phi(p_0) .$$

Since $\phi(p_0)$ is contained in the center of $A'' \times_{\alpha''} G$ and α'' acts centrally freely, there exists a projection q_0 in the center of A'' such that $\phi(p_0) = \pi_\alpha(q_0)$. Let

W_s be an operator on $l^2(G, H)$ to H such that $W_s\xi = \xi(s^{-1})$ for each s in G and ξ in $l^2(G, H)$. Put for every x in $B(l^2(G, H))$, $E(x) = W_e x W_e^*$ where e is the identity of G. We also denote the restriction of E to $A'' \times_{\alpha''} G$ by E. Then $E: A'' \times_{\alpha''} G \to A''$ is a faithful normal positive linear map. Let $\{p_i\}_{i\in I}$ be an approximate unit of J. Then p_0 is the least upper bound of $\{p_i\}$. Since p_i exists in J for every i in I, $E(\phi(p_i))$ is contained in A for every i in I. Since $\phi(p_i) \leq \phi(p_0)$ for every i in I, we get for every i in I

$$E(\phi(p_i))q_0 = E(\phi(p_i)\pi_\alpha(q_0)) = E(\phi(p_i)\phi(p_0)) = E(\phi(p_i)).$$

Thus we have for every i in I

$$\pi_\alpha(E(\phi(p_i))) = \pi_\alpha(E(\phi(p_i)))\phi(p_0).$$

Therefore $\pi_\alpha(E(\phi(p_i)))$ is contained in $\phi(J) \cap \pi_\alpha(A)$ for every i in I. Since A is G-simple, $\phi(J) \cap \pi_\alpha(A)$ is $\{0\}$ or $\pi_\alpha(A)$. Since J is nonzero,

$$\phi(J) \cap \pi_\alpha(A) = \pi_\alpha(A).$$

So we have $J = A \times_\alpha G$. The converse is an immediate consequence of Lemma 1.

Corollary 3. *Let G be a discrete group and let (A, G, α) be a C^*-dynamical system in which G is a central shift. Then A is G-simple if and only if the reduced crossed proudct $A \times_{\alpha r} G$ is simple.*

Proof. Let J be a nonzero norm closed two-sided ideal of the reduced crossed product $A \times_{\alpha r} G$. There exists a projection p in the center of the W^*-crossed product $A'' \times_{\alpha''} G$ such that

$$\overline{J}^{\sigma w} = (A'' \times_{\alpha''} G)p$$

where $\overline{J}^{\sigma w}$ denotes the σ-weak closure of J. We then proceed with the remaining part of this proof in the similar manner as the proof of Theorem 2.

Corollary 4. *Let (A, G, α) be a C^*-dynamical system and let G be a discrete group. Assume that G is a central shift in (A, G, α). Then A is G-prime if and only if the C^*-crossed product $A \times_\alpha G$ is prime.*

Proof. Let I_1 and I_2 be nonzero α-invariant closed two-sided ideals of A. Since $A \times_\alpha G$ is prime, $(I_1 \times_\alpha G) \cap (I_2 \times_\alpha G)$ is a nonzero two-sided ideal of $A \times_\alpha G$. Since G is discrete, there exists a conditional expectation $P: l^1(G, A) \to A$ defined by $P(x) = x(e)$ for all x in $l^1(G, A)$, where e is the identity of G. Then P can be extended to a conditional expectation of $A \times_\alpha G$ onto A. If we consider the restriction of P to $I_k \times_\alpha G$ for each $k = 1$ and 2, then $P(x^*x)$ is a nonzero element in $I_1 \cap I_2$ for every nonzero element x in $(I_1 \times_\alpha G) \cap (I_2 \times_\alpha G)$. Thus $I_1 \cap I_2 \neq \{0\}$.

The converse can be proved by imitating the proof of Theorem 2.

Under the same hypothesis of Corollary 4 we can also show that A is G-prime if and only if the reduced crossed product $A \times_{\alpha r} G$ is prime.

Remark 5. Let (A, G, α) be a C^*-dynamical system and let G be a discrete group. If G is a central shift in (A, G, α) then α is properly outer. For abelian C^*-algebras [3] and AF-algebras [2], it was shown that if α is properly outer then G-simplicity of A implies simplicity of $A \times_\alpha G$. It would be nice if we can extend this statement for more general C^*-algebras.

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