SIMPLICITY OF CROSSED PRODUCTS OF $C^*$-ALGEBRAS

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ABSTRACT. Let $(A, G, \alpha)$ be a $C^*$-dynamical system and let $G$ be a discrete group. When $G$ is a central shift in $(A, G, \alpha)$, we show that $A$ is $G$-simple (resp. $G$-prime) if and only if the $C^*$-crossed product $A \times_\alpha G$ is simple (resp. prime).

1. Introduction

Let $(A, G, \alpha)$ be a $C^*$-dynamical system. Our aim is to continue the investigation of the relationship between the property of the $C^*$-dynamical system $(A, G, \alpha)$ and the ideal structure of the corresponding $C^*$-crossed product $A \times_\alpha G$. This problem first appeared in [6] and has been studied in [2, 3, 4, etc.]. Olesen and Pedersen [7] gave the necessary and sufficient condition of simplicity of $C^*$-crossed products by locally compact abelian groups. When $G$ is a discrete group and $A$ is an AF-algebra, Elliott [2] showed that if $\alpha$ is properly outer and $A$ is $G$-simple then the reduced product $A \times_{\alpha_r} G$ is simple. Later, Kawamura and Tomiyama [3] obtained the same result when $A$ is an abelian $C^*$-algebra. In this paper we study simplicity of the $C^*$-crossed product $A \times_\alpha G$ for a general $C^*$-algebra when $G$ is a discrete group.

Let $(A, G, \alpha)$ be a $C^*$-dynamical system and $G$ be a discrete group. Let $A''$ be the enveloping von Neumann algebra of a $C^*$-algebra $A$. Then the action $\alpha: g \mapsto \alpha_g$ induces the action $\alpha'': g \mapsto \alpha''_g$ on $A''$. Then $(A'', G, \alpha'')$ becomes a $W^*$-dynamical system. It is said that $G$ is a central shift in $(A, G, \alpha)$ if there exists a central projection $p$ in $A''$ such that $\sum_{g \in G} \alpha''_g(p) = 1$ and $\alpha''_g(p)p = 0$ for every $g \neq e$ in $G$, where $e$ is the identity of $G$ (see [1]). When $G$ is a central shift in $(A, G, \alpha)$, we show that $A$ is $G$-simple if and only if the $C^*$-crossed product $A \times_\alpha G$ is simple.

2. Main result

Let $(M, G, \alpha)$ be a $W^*$-dynamical system and $M \subset B(H)$ for a Hilbert space $H$. The $W^*$-crossed product $M \times_\alpha G$ is the von Neumann algebra on...
generated by \( \{\pi_\alpha(x), \lambda_g | x \in M, g \in G\} \), where
\[
\begin{align*}
(\pi_\alpha(x)\xi)(s) &= \alpha_{s^{-1}}(x)\xi(s), \\
(\lambda_g \xi)(s) &= \xi(g^{-1}s)
\end{align*}
\]
for every \( x \in M, s, g \in G \), and \( \xi \) in \( L^2(G, H) \).

**Lemma 1.** Let \((A, G, \alpha)\) be a \( C^* \)-dynamical system. Let \( I \) be an \( \alpha \)-invariant norm closed two-sided ideal of \( A \). Then \( I \times_\alpha G \) is a norm closed two-sided ideal of \( A \times_\alpha G \).

**Proof.** Let \((\pi \times \lambda)\) be the universal representation of \( A \times_\alpha G \) induced by some covariant representation \((\pi, \lambda, H)\) of the \( C^* \)-dynamical system \((A, G, \alpha)\), and let \((A \times_\alpha G)^{''}\) be the enveloping von Neumann algebra of \( A \times_\alpha G \). Let \( I \) be an \( \alpha \)-invariant norm closed two-sided ideal of \( A \). Let \( p \) be a projection in the center of \( \overline{\pi(A)}^{\sigma w} \) such that
\[
\overline{\pi(I)}^{\sigma w} = \overline{\pi(A)}^{\sigma w} p
\]
where \( \overline{()^{\sigma w}} \) means the \( \sigma \)-weak closure of \( () \). It is clear that \( p \) is contained in \((A \times_\alpha G)^{''}\). Since \( I \) is \( \alpha \)-invariant, we have that \( \lambda_g p \lambda_g^* = p \) for every \( g \) in \( G \). Then we put for every \( f \) in \( L^1(G) \)
\[
\lambda_f = \int_G f(g) \lambda_g \, dg
\]
where \( dg \) is the Haar measure on \( G \). For \( x \) in \( A \) and \( f \) in \( L^1(G) \) we obtain
\[
\pi(x)\lambda_f p = \pi(x)p\lambda_f = p\pi(x)\lambda_f.
\]
Since \((A \times_\alpha G)^{''}\) is generated by \( \{\pi(x)\lambda_f | x \in A, f \in L^1(G)\} \), \( p \) is contained in the center of \((A \times_\alpha G)^{''}\). Also, since \((I \times_\alpha G)^{''}\) is generated by \( \{\pi(x)\lambda_f | x \in I, f \in L^1(G)\} \), we get
\[
(\pi \times \lambda)(A \times_\alpha G)^{''} p = (A \times_\alpha G)^{''} = (I \times_\alpha G)^{''}.
\]
It follows that \( I \times_\alpha G \) is a norm closed two-sided ideal of \( A \times_\alpha G \).

**Theorem 2.** Let \((A, G, \alpha)\) be a \( C^* \)-dynamical system and let \( G \) be a discrete group. Let \( G \) be a central shift in \((A, G, \alpha)\). Then \( A \) is \( G \)-simple if and only if the \( C^* \)-crossed product \( A \times_\alpha G \) is simple.

**Proof.** First suppose that \( A \) is \( G \)-simple. Let \( A'' \) be the enveloping von Neumann algebra of \( A \), and let \((A'', G, \alpha'')\) be the \( W^* \)-dynamical system induced by the \( C^* \)-dynamical system \((A, G, \alpha)\). Since \( G \) is a central shift in \((A, G, \alpha)\), by [1] there exists an *-isomorphism \( \phi \) from the enveloping von Neumann algebra \((A \times_\alpha G)^{''}\) of \( A \times_\alpha G \) onto the \( W^* \)-crossed product \( A'' \times_{\alpha''} G \). Let \( J \) be a nonzero norm closed two-sided ideal of \( A \times_\alpha G \). There exists a projection \( p_0 \) in the center of \((A \times_\alpha G)^{''}\) such that
\[
\overline{J}^{\sigma w} = (A \times_\alpha G)^{''} p_0
\]
where \( \overline{J}^{\sigma w} \) denotes the \( \sigma \)-weak closure of \( J \). Then we have
\[
\phi(\overline{J}^{\sigma w}) = (A'' \times_{\alpha''} G)\phi(p_0).
\]
Since \( \phi(p_0) \) is contained in the center of \( A'' \times_{\alpha''} G \) and \( \alpha'' \) acts centrally freely, there exists a projection \( q_0 \) in the center of \( A'' \) such that \( \phi(p_0) = \pi_\alpha(q_0) \). Let
$W_s$ be an operator on $l^2(G, H)$ to $H$ such that $W_s\xi = \xi(s^{-1})$ for each $s$ in $G$ and $\xi$ in $l^2(G, H)$. Put for every $x$ in $B(l^2(G, H))$, $E(x) = W_x W_e^*$ where $e$ is the identity of $G$. We also denote the restriction of $E$ to $A'' \times_{\alpha''} G$ by $E$. Then $E: A'' \times_{\alpha''} G \to A''$ is a faithful normal positive linear map. Let $\{p_i\}_{i \in I}$ be an approximate unit of $J$. Then $p_0$ is the least upper bound of $\{p_i\}$. Since $p_i$ exists in $J$ for every $i$ in $I$, $E(\phi(p_i))$ is contained in $A$ for every $i$ in $I$. Since $\phi(p_i) \leq \phi(p_0)$ for every $i$ in $I$, we get for every $i$ in $I$
$$E(\phi(p_i))q_0 = E(\phi(p_i)\pi_\alpha(q_0)) = E(\phi(p_i)\phi(p_0)) = E(\phi(p_i)).$$
Thus we have for every $i$ in $I$
$$\pi_\alpha(E(\phi(p_i))) = \pi_\alpha(E(\phi(p_i)))\phi(p_0).$$
Therefore $\pi_\alpha(E(\phi(p_i)))$ is contained in $\phi(J) \cap \pi_\alpha(A)$ for every $i$ in $I$. Since $A$ is $G$-simple, $\phi(J) \cap \pi_\alpha(A)$ is $\{0\}$ or $\pi_\alpha(A)$. Since $J$ is nonzero,
$$\phi(J) \cap \pi_\alpha(A) = \pi_\alpha(A).$$
So we have $J = A \times_\alpha G$. The converse is an immediate consequence of Lemma 1.

**Corollary 3.** Let $G$ be a discrete group and let $(A, G, \alpha)$ be a C*-dynamical system in which $G$ is a central shift. Then $A$ is $G$-simple if and only if the reduced crossed product $A \times_{\alpha} G$ is simple.

**Proof.** Let $J$ be a nonzero norm closed two-sided ideal of the reduced crossed product $A \times_{\alpha} G$. There exists a projection $p$ in the center of the $W^*$-crossed product $A'' \times_{\alpha''} G$ such that
$$\overline{J}^{\sigma_w} = (A'' \times_{\alpha''} G)p$$
where $\overline{J}^{\sigma_w}$ denotes the $\sigma$-weak closure of $J$. We then proceed with the remaining part of this proof in the similar manner as the proof of Theorem 2.

**Corollary 4.** Let $(A, G, \alpha)$ be a C*-dynamical system and let $G$ be a discrete group. Assume that $G$ is a central shift in $(A, G, \alpha)$. Then $A$ is $G$-prime if and only if the C*-crossed product $A \times_{\alpha} G$ is prime.

**Proof.** Let $I_1$ and $I_2$ be nonzero $\alpha$-invariant closed two-sided ideals of $A$. Since $A \times_{\alpha} G$ is prime, $(I_1 \times_{\alpha} G) \cap (I_2 \times_{\alpha} G)$ is a nonzero two-sided ideal of $A \times_{\alpha} G$. Since $G$ is discrete, there exists a conditional expectation $P: l^1(G, A) \to A$ defined by $P(x) = x(e)$ for all $x$ in $l^1(G, A)$, where $e$ is the identity of $G$. Then $P$ can be extended to a conditional expectation of $A \times_{\alpha} G$ onto $A$. If we consider the restriction of $P$ to $I_k \times_{\alpha} G$ for each $k = 1$ and 2, then $P(x^*x)$ is a nonzero element in $I_1 \cap I_2$ for every nonzero element $x$ in $(I_1 \times_{\alpha} G) \cap (I_2 \times_{\alpha} G)$. Thus $I_1 \cap I_2 \neq \{0\}$.

The converse can be proved by imitating the proof of Theorem 2.

Under the same hypothesis of Corollary 4 we can also show that $A$ is $G$-prime if and only if the reduced crossed product $A \times_{\alpha} G$ is prime.

**Remark 5.** Let $(A, G, \alpha)$ be a C*-dynamical system and let $G$ be a discrete group. If $G$ is a central shift in $(A, G, \alpha)$ then $\alpha$ is properly outer. For abelian C*-algebras [3] and AF-algebras [2], it was shown that if $\alpha$ is properly outer then $G$-simplicity of $A$ implies simplicity of $A \times_{\alpha} G$. It would be nice if we can extend this statement for more general C*-algebras.
References


