HOMOGENIZED $\mathfrak{sl}(2)$

LIEVEN LE BRUYN AND S. P. SMITH

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Abstract. This note studies a special case of Artin's projective geometry (Geometry of quantum planes, MIT, preprint, 1990) for noncommutative graded algebras. It is shown that (most of) the line modules over the homogenization of the enveloping algebra $U(\mathfrak{sl}(2, \mathbb{C}))$ are in bijection with the lines lying on the quadrics that are the (closures of the) conjugacy classes in $\mathfrak{sl}(2, \mathbb{C})$. Furthermore, these line modules are the homogenization of the Verma modules for $\mathfrak{sl}(2, \mathbb{C})$.

1. LEVEL QUADRICS FOR $\mathfrak{sl}(2, \mathbb{C})$

Throughout we will write $\mathfrak{g} = \mathbb{C} \mathfrak{e} \oplus \mathbb{C} \mathfrak{f} \oplus \mathbb{C} \mathfrak{h}$ and define a vector space isomorphism $\mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{g}$ by

$$
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix} \to e, 
\begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix} \to f, 
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix} \to h.
$$

We transfer the Lie bracket on $\mathfrak{sl}(2, \mathbb{C})$ to $\mathfrak{g}$ giving

$$
[e, f] = h, 
[h, e] = 2e, 
[h, f] = -2f.
$$

The cone of nilpotent elements in $\mathfrak{sl}(2, \mathbb{C})$ is the variety defined by the quadratic relation $det = 0$ where $det$ is the determinant function on $\mathfrak{sl}(2, \mathbb{C})$. The conjugacy classes of semisimple elements in $\mathfrak{sl}(2, \mathbb{C})$ are the level surfaces $det + \lambda^2 = 0$, where $\lambda \in \mathbb{C}^*$; in particular, this surface is the conjugacy class of the element $\left(\begin{array}{cc}
\lambda & 0 \\
0 & -\lambda
\end{array}\right)$.

Transferring this to $\mathfrak{g}$ via the above isomorphism, it follows that the determinant function on $\mathfrak{g}$ is given by $det = -h^*2 - e^*f^*$, where $e^*, f^*, h^*$ are the dual bases to $e, f, h$, respectively. Hence the nilpotent cone (resp. the conjugacy class of the element $\lambda h$) is given by the quadric surface $-h^*2 - e^*f^* = 0$ (resp. $= -\lambda^2$) in $\mathfrak{g}$.

We identify $\mathfrak{g}$ with $\mathfrak{g}^*$ through the Killing form induced by the nondegenerate pairing

$$
\mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C}) \to \mathbb{C}, 
(x, y) \to Tr(x.y).
$$

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Transferring this to \( \mathfrak{g} \) gives the identifications \( e = f^* \), \( f = e^* \), and \( h = 2h^* \). Under this identification the nilpotent cone (resp. the conjugacy class of \( \lambda h \)) is given by the equation \( \det = -\frac{1}{2} h^2 - ef = 0 \) (resp. \( = -\lambda^2 \)).

We may homogenize the defining equations with respect to a new variable \( t \) and thus consider the following pencil of quadrics in \( \mathbb{P}^3 \):

\[
Q(\delta) = \mathcal{V}(\det + \delta^2 t^2) \quad \text{for all} \quad \delta \in \mathbb{P}^1.
\]

The base locus of this pencil is the conic \( \mathcal{V}(t^2, -h^2 - 4ef) \) in the plane at infinity. The only singular quadrics in this pencil are \( Q(0) \) and \( Q(\infty) \) (the plane at infinity twice). If we identify \( \mathfrak{g} \) with the affine open piece \( t = 1 \), then the intersection of \( Q(0) \) with \( \mathfrak{g} \) is the cone of nilpotent elements. If \( \delta \neq 0, \infty \) then \( Q(\delta) \) is smooth and its intersection with \( \mathfrak{g} \) is a conjugacy class of semisimple elements, the conjugacy class of \( \delta h \).

If \( Q(\delta) \) is smooth then \( Q(\delta) \cong \mathbb{P}^1 \times \mathbb{P}^1 \), and there are two families of lines lying on \( Q(\delta) \). On the other hand, \( Q(0) \) is singular and contains only one family of lines; see [4, Chapter II, §6]. Our first objective is to characterize in terms of \( \mathfrak{sl}(2, \mathbb{C}) \) those lines in \( \mathbb{P}^3 \) that lie on some quadric \( Q(\delta) \).

Any two-dimensional Lie subalgebra of \( \mathfrak{g} \) is a Borel subalgebra. A standard basis for a Borel subalgebra is an ordered basis \( (E, H) \) such that \([H, E] = 2E\). Any standard basis for a Borel subalgebra may be extended to a standard triple \((E, F, H)\) that is a triple of linearly independent elements such that

\[
\]

The standard triples form a single orbit under conjugation by \( \text{GL}(2) \). If \( b \) is a Borel subalgebra with standard basis \((E, H)\) then \( \lambda \in \mathbb{C} \) determines \( f_\lambda \in b^* \) by requiring that \( f_\lambda(E) = 0 \) and \( f_\lambda(H) = \lambda \). Notice that \( f_\lambda \) depends only on \( \lambda \) and not on the choice of the standard basis, since any other standard basis for \( b \) is of the form \( \nu E, H + \mu E \).

For each pair \((b, \lambda)\) we define a line in \( \mathbb{P}^3 \):

\[
l_{b, \lambda} = \mathcal{V}(E, H - \lambda t),
\]

where \((E, H)\) is standard basis for \( b \). Notice that this line does not depend on the choice of a standard basis for \( b \). If we identify the plane at infinity with \( \mathbb{P}(\mathfrak{g}) \) and the affine open piece as before with \( \mathfrak{g} \), then the following two facts are easily verified:

1. \( l_{b, \lambda} \cap \mathbb{P}(\mathfrak{g}) = [b, b] = \mathbb{C}E \),
2. \( l_{b, \lambda} \cap \mathfrak{g} = \frac{1}{2} \lambda H + \mathbb{C}E \), where \((E, H)\) is a standard basis for \( b \).

In particular, \( l_{b, \lambda} \) is a line on the quadric \( Q(\frac{1}{2} \lambda) \). In fact,

**Theorem 1.** The lines that lie on the quadrics \( Q(\delta) = \mathcal{V}(\det + \delta^2 t^2) \) for some \( \delta \in \mathbb{P}^1 \) are

1. the lines at infinity and
2. lines \( l_{b, \lambda} \) for a Borel subalgebra \( b \) and \( \lambda \in \mathbb{C} \).

**Proof.** The first case corresponds to lines on \( Q(\infty) \). Suppose \( \delta \neq 0 \). Then \( Q(\delta) \cap \mathfrak{g} \) is the conjugacy class of \( \begin{pmatrix} \delta & 0 \\ 0 & -\delta \end{pmatrix} \). If \( l \) is a line lying on \( Q(\delta) \), then we can choose a basis for \( \mathfrak{sl}(2, \mathbb{C}) \) such that \( l = \{x + \nu y | \nu \in \mathbb{C}\} \), where \( x = \begin{pmatrix} \delta & 0 \\ 0 & -\delta \end{pmatrix} \) and \( y \) is some other element of \( \mathfrak{sl}(2, \mathbb{C}) \). Write \( y = \begin{pmatrix} 1 & b \\ c & d \end{pmatrix} \). Since \( \det(x + \nu y) = \det(x) \) for all \( \nu \), a calculation shows that \( \det(y) = 0 \) and \( a = 0 \).
Thus \( y \) is (up to scalar multiples) either \( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) of \( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \). In the first case \( l = l_{\delta, 2\delta} \) with \( b \) having standard basis \( y, x \). In the second case \( l = l_{\delta, -2\delta} \) with \( b \) having standard basis \( y, -x \).

Suppose that \( \delta = 0 \), so that \( Q(\delta) \) is the nilpotent cone. We can choose a basis for \( \mathfrak{sl}(2, \mathbb{C}) \) such that \( l = \{ x + \nu y | \nu \in \mathbb{C} \} \), where \( x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) and \( y = \begin{pmatrix} 1 & b \\ c & d \end{pmatrix} \). Since \( \det(x + \nu y) = 0 \) for all \( \nu \), a calculation shows that \( \det(y) = 0 \) and \( c = 0 \) (whence \( a = 0 \) also). Thus \( y \) is a scalar multiple of \( x \), so the line is \( l = \mathbb{C}x = l_{b, \lambda} \) having standard basis \( x, H \), where \( H \) satisfies \( [H, x] = 2x \) (such an \( H \) does exist since every nonzero nilpotent element belongs to some standard triple).

Conversely, we have already seen that \( l_{b, \lambda} \) lies on the quadric \( Q(\frac{1}{2} \lambda) \).

If \( \delta \neq 0 \), \( \infty \) then the two rulings on \( Q(\delta) \) are given by \( \{ l_{b, 2\delta} | b \text{ is a Borel} \} \) and \( \{ l_{b, -2\delta} | b \text{ is a Borel} \} \).

2. The quantum space of \( \mathfrak{sl}(2, \mathbb{C}) \)

Let \( A \) denote the homogenization of the enveloping algebra of \( \mathfrak{g} \) with respect to a central variable \( t \). That is, \( A = \mathbb{C}[e, f, h, t] \) with defining equations

\[
\begin{align*}
e f - fe &= ht, \\
he - eh &= 2et, \\
hf - fh &= -2ft; \\
et - te &= ft - tf = ht - th = 0.
\end{align*}
\]

Notice that \( A/A(t - 1) \cong U(g) \), the enveloping algebra of \( g \), and \( A/At \cong \mathbb{C}[e, f, h] = S(g) \), the symmetric algebra on \( g \). From these facts one deduces that \( A \) has Hilbert series \( (1 - t)^{-4} \), is a positively graded Noetherian domain, a maximal order, Auslander-regular of dimension \( 4 \), and satisfies the Cohen-Macaulay property, see, e.g., [5-7]. Moreover, the center of \( A \) is \( \mathbb{C}[\Omega, t] \), where \( \Omega = h^2 + 2ef + 2fe \) is the Casimir element.

Artin [1] associates to any regular algebra \( R \) its quantum space \( \text{Proj}(R) \), which is by definition the quotient-category of all finitely generated graded left \( R \)-modules by the full Serre subcategory of the finite length modules. We will denote the quantum space associated to \( A \) by \( Q(g) \). We want to characterize the linear subspaces in \( Q(g) \). There are three types to consider:

1. A plane module \( \mathcal{P} \) is a cyclic module with Hilbert series \( (1 - t)^{-3} \).
2. A line module \( \mathcal{L} \) is a cyclic module with Hilbert series \( (1 - t)^{-2} \).
3. A point module \( \mathcal{P} \) is a cyclic module with Hilbert series \( (1 - t)^{-1} \).

As in [6] these modules (together with \( A \) and the trivial module \( A/Ae + Af + Ah + At \) are precisely the Cohen-Macaulay modules of multiplicity one. We will associate to each of them a linear subspace in ordinary \( \mathbb{P}^3 \).

As \( \dim(A_1) = 4 \) and the homogeneous degree 1 component of a plane (resp. line, point) module is 3 (resp. 2, 1) one can find \( a \in A_1 \) (resp. \( a, b \in A_1 \), resp. \( a, b, c \in A_1 \)) and surjections

\[
A/Aa \to \mathcal{P} \quad (\text{resp. } A/(Aa + Ab) \to \mathcal{L}, \quad \text{resp. } A/(Aa + Ab + Ac) \to \mathcal{P})
\]

(in fact we will see shortly that these maps have to be isomorphisms). Hence to each plane (resp. line, point) module we can associate a plane (resp. line, point) in \( \mathbb{P}^3 = \mathbb{P}(A_1^*) \), namely, \( \mathcal{V}(a) \) (resp. \( \mathcal{V}(a, b) \), resp. \( \mathcal{V}(a, b, c) \)). We will now determine which linear subspaces of \( \mathbb{P}^3 \) arise in this way.
We observe that there is a dichotomy in the problem. As a linear subspace module is critical (because it is of multiplicity 1) it follows from [6] that \( t \) either kills the module or acts faithfully. The first case gives a linear subspace module over the commutative polynomial ring \( A/At \approx S(g) \), i.e., we get a linear subspace in \( \mathbb{P}^2 = \mathcal{V}(t) = \mathbb{P}(g) \). In the second case we can form \( M = M/(t-1)M \), which is a filtered \( U(g) \)-module \( M_0 \subset M_1 \subset \cdots \), where \( \dim(M_i) \) is equal to \( a \) (if \( M \) is a point module), \( i+1 \) (if \( M \) is a line module), or \( \frac{1}{2}(i+1)(i+2) \) (if \( M \) is a plane module). Moreover, this process can be reversed, namely, \( M \approx \bigoplus M_i t^i \) (see [3] or [7]).

**Proposition 1.**

1. Every plane \( \mathcal{V}(a) \) in \( \mathbb{P}^3 \) determines a plane module \( S \).
2. The points at infinity and the origin \( (0, 0, 0, 1) \) are the only points in \( \mathbb{P}^3 \) that determine a point module \( S \).

**Proof.** Since \( A \) is a domain, \( A/Aa \) has Hilbert series \((1 - t)^{-3}\) for every nonzero \( a \in A_1 \). Hence the surjection \( A/Aa \to S \) is an isomorphism.

From the dichotomy remark it follows that a point-module either corresponds to a point on the plane at infinity (when it is killed by \( t \)) or corresponds to the 1-dimensional \( g \)-module, whence it corresponds to the origin in \( g \) (which is bing identified with the complement to the plane at infinity). In the first case we can take \( a = t \) and let \( b, c \) determine the point in \( \text{Proj}(A/At) = \mathbb{P}^2 \). Hence the Hilbert series of \( A/At + Ab + Ac \) is \( (1 - t)^{-1} \) so \( S = A/At + Ab + Ac \). In the second case \( A/Ae + Af + Ah \approx \mathbb{C}[t] \), which also has the right Hilbert series. \( \square \)

Still the situation concerning point-modules is slightly more subtle. The set of points obtained from point modules is called the point variety, and it may be described by multilinearizing the defining relations as in [2] or [8]; that is, the point-variety is the zero set of the \( 4 \times 4 \) minors of either of the following two matrices:

\[
\begin{pmatrix}
-f & e & 0 & -h \\
h & 0 & -e & -2e \\
0 & h & -f & 2f \\
-t & 0 & 0 & e \\
0 & -t & 0 & f \\
0 & 0 & -t & h \\
\end{pmatrix}
\]

or

\[
\begin{pmatrix}
f & -h - 2t & 0 & t & 0 & 0 \\
-e & 0 & -h + 2t & 0 & t & 0 \\
-t & e & f & 0 & 0 & t \\
0 & 0 & 0 & -e & -f & -h \\
\end{pmatrix}
\]

**Proposition 2.** The ideal determining the point-modules of \( A \) in \( \mathbb{P}^3 \) is

\[t(h^2 + 4ef)(e, f, h), te, tf, th).\]

Hence the conic at infinity \( \mathcal{V}(t, h^2 + 4ef) \) is an embedded component of the point-variety.

So the base locus of the pencil of quadrics described in §1 appears here as the embedded component of the point-variety for \( A \). Let us now describe the line-modules of \( Q(g) \).
Theorem 2. The lines in \( \mathbb{P}^3 \) determining line modules are precisely the lines in the pencil of quadrics \( Q(\delta) = \mathcal{V}(\det + \delta^2 t^2) \) for \( \delta \in \mathbb{P}^1 \).

Proof. As in the case of point-modules, the lines in the plane at infinity are already accounted for. Hence we have to prove that any line module of \( A \) is of the form

\[ \mathscr{L} \cong A/(AE + A(H - \lambda t)), \]

where \( E, H \) is a standard basis for a Borel subalgebra \( \mathfrak{b} \) and \( \lambda \in \mathbb{C} \).

There is a surjection \( A/Aa + Ab \rightarrow \mathscr{L} \) for some \( a, b \in A_1 \) since \( \dim L_1 = 2 \). Clearly, we can change \( a, b \) if necessary so that \( a = y, b = z + \lambda t \) with \( y, z \in \mathfrak{g} \). Let \( x, y, z \) be a basis of \( \mathfrak{g} \). Then the following seven linearly independent elements in \( A_2 \) belong to \( Aa + Ab \):

\[ xy, yy, ty = yt, x(z + \lambda t), y(z + \lambda t), z(z + \lambda t), t(z + \lambda t) = (z + \lambda t)t. \]

As \( \dim \mathscr{L}_2 = 3 \) and \( \dim(A_2) = 10 \) these elements must be a basis for \( (Aa + Ab)_2 \). Note that both \( yz = y(z + \lambda t) - \lambda yt \) and \( yz \) belong to this space. Hence \( yz - yz \) can be written as a linear combination of the seven elements. If \( \lambda \neq 0 \) then only \( ty \) can occur with nonzero coefficient, and if \( \lambda = 0 \) so might \( tz \). At any rate \( b = C y \oplus Cz \) is a two-dimensional Lie subalgebra of \( \mathfrak{g} \) and hence is a Borel subalgebra.

But then \( A/(Aa + Ab) \) is the homogenization of \( U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C} f \) for some \( f \in \mathfrak{g}^* \) such that \( f|_{[b, b]} = 0 \) and hence has as its Hilbert series \( (1 - t)^{-2} \). Therefore, \( A/Aa + Ab = \mathscr{L} \) and \( \mathcal{V}(a, b) = I_{b, \lambda} \) as claimed. \( \square \)

3. SOME COMMENTS

The line module associated to \( I_{b, \lambda} \) will be denoted by \( M(b, \lambda) \). By Theorem 2 we have \( M(b, \lambda) \cong A/AE + A(H - \lambda t) \), and hence \( \mathcal{V}(a, b) \) is the homogenization of the Verma module \( M_b(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C} \lambda \) of highest weight \( \lambda \). The \((n + 1)\)-dimensional simple \( U(\mathfrak{sl}(2, \mathbb{C}))\)-modules will be denoted \( V(n) \). For each Borel \( \mathfrak{b} \) there is a short exact sequence \( 0 \rightarrow M_b(-n - 2) \rightarrow M_b(n) \rightarrow V(n) \rightarrow 0 \). Taking homogenized modules, there is a corresponding short exact sequence \( 0 \rightarrow M(b, -n - 2) \rightarrow M(b, n) \rightarrow F(n) \rightarrow 0 \) where \( F(n) \) is a certain fat point of multiplicity \( n + 1 \); a fat point module \([1]\) is a 1-critical module generated in degree 0, having constant Hilbert series, and a fat point is a module that is an \( A \)-module that is equivalent to a fat point module in the sense that they give isomorphic objects of \( \text{Proj}(A) \).

This is reminiscent of some of the results on the Sklyanin algebra in \([9]\). Homogenized \( \mathfrak{sl}(2, \mathbb{C}) \) shares some other common features with the Sklyanin algebra; for example annihilators of line modules behave in a similar way. We leave the details to the interested reader.

The quantum space of any three-dimensional Lie algebra has similar properties. Let us briefly sketch the case of the three-dimensional Lie algebra \( \mathfrak{h} = \mathbb{C} x \oplus \mathbb{C} y \oplus \mathbb{C} z \) with \( z \) central and \([x, y] = z \). The corresponding algebra \( H(\mathfrak{h}) = \mathbb{C}[x, y, z, t] \) with relations \( xy - yx = zt \) and \( z \) and \( t \) central; the point variety in \( Q(\mathfrak{h}) = \text{Proj}(H(\mathfrak{h})) \) is determined by the ideal \( tz(z, t) \) and so consists of the union of the plane \( \mathcal{V}(z) \) and the plane at infinity \( \mathcal{V}(t) \); their intersection is an embedded component.

The line modules of \( H(\mathfrak{h}) \) are precisely the lines in the pencil of planes \( P(\delta) = \mathcal{V}(z + \delta t) \) for \( \delta \in \mathbb{P}^1 \), which has as its base locus the embedded component \((z, t)\).
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Department of Wiskunde, Universitaire Instelling Antwerpen B-2610 Wilrijk, Belgium
E-mail address: lebruyn@ccu.uia.ac.be

Department of Mathematics, University of Washington, Seattle, Washington 98195
E-mail address: smith@math.washington.edu