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MANIFOLDS WITH PINCHED RADIAL CURVATURE

YOSHIROH MACHIGASHIRA

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ABSTRACT. We generalize Toponogov's theorem to the context of radial curvature and obtain corresponding generalizations of classical sphere theorems.

0. INTRODUCTION

Throughout this paper, let M be a connected complete Riemannian n-manifold. It is interesting to investigate the relation between curvature and topology of Riemannian manifolds as exemplified in sphere theorems. The classical sphere theorem is due to Rauch, Berger, and Klingenberg, and it states that if M is simply connected and the sectional curvature K_M of M satisfies $\frac{1}{4} < K_M \leq 1$, then M is homeomorphic to the n-dimensional sphere S^n (see [1, Theorem 6.1]). The sectional curvature of M is a function on a bundle $G_{n,2}(M)$ of Grassmannians of the tangent planes over M. For a fixed point p of M, we consider the restriction K_p of K_M to certain subspaces of $G_{n,2}(M)$ called the radial planes from p. This notion was first introduced by Klingenberg [5].

Let us introduce some definitions. We agree that geodesics are parameterized by arclength. Fix a point $p \in M$ and take any geodesic $\gamma\colon [0, l] \to M$ emanating from p. By definition, a plane σ in $T_{\gamma(t)}M$ (i.e., a 2-dimensional subspace of $T_{\gamma(t)}M$) for $t \in [0, l]$ is a radial plane from p along γ if $\dot{\gamma}(t)$ is contained in σ. We call a radial plane from p along some geodesic a radial plane from p. We put

$$\Sigma(p) := \{\sigma \mid \text{a radial plane from } p\},$$
$$\Sigma^a(p) := \{\sigma \mid \text{a radial plane from } p \text{ along some geodesic } \gamma \text{ with } L(\gamma) \leq a\},$$
$$\Sigma^{\min}(p) := \{\sigma \mid \text{a radial plane from } p \text{ along some minimal geodesic } \gamma\},$$

where $a \in [0, \infty)$ is any number and $L(\gamma)$ the length of γ. Note that any plane in T_pM is contained in $\Sigma(p)$, $\Sigma^a(p)$, and $\Sigma^{\min}(p)$. By the radial curvature K_p from p (resp. the a-radial curvature K_p^a from p, the min-radial curvature $K_p^{\min}$ from p), we mean the restriction of the sectional curvature function to $\Sigma(p)$ (resp. $\Sigma^a(p)$, $\Sigma^{\min}(p)$).

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Then clearly we see that under suitable conditions on K_p, Rauch's comparison theorem holds for Jacobi fields along geodesics through p and that Bishop-Gromov's comparison theorem holds for metric balls around p. Moreover, Toponogov's comparison theorem holds for the edge angles at points q_1 and q_2 of minimal geodesic triangles with vertices at p, q_1, and q_2 (see Proposition 1.1). Thus it is natural to ask if some sphere theorems hold with assumption on radial curvature. Also in the case where M is noncompact and p is a pole of M, Greene and Wu have obtained many results related to the radial curvature from p (see [3]).

First of all, Klingenberg showed

Theorem (Klingenberg [5]). *Let M be simply-connected. If there exists a point $p \in M$ such that $\frac{1}{4} < K_p^{2\pi} \leq 1$, then M has the homotopy type of S^n.*

The above theorem implies that if $n \neq 3$ then M is homeomorphic to S^n. In case $n = 3$, Hamilton proved that a closed 3-manifold with positive Ricci-curvature admits a Riemannian metric of positive constant curvature [4]. However, it is not known that the condition $\frac{1}{4} < K_p^{2\pi} \leq 1$ implies the positivity of the Ricci-curvature of M. The purpose of this paper is to show by direct geometric methods that M is homeomorphic to S^n under the same assumption for all dimensions. It is natural to consider whether M is diffeomorphic to S^n under stronger restriction of the radial curvature. This question is solved affirmatively by assuming a lower bound of K_M.

Theorem A. *Let M be simply-connected. If there exists a point $p \in M$ such that*

$$\tfrac{1}{4} < K_p^{2\pi} \leq 1,$$

then M is homeomorphic to S^n.

Theorem B. *For given $\kappa \geq 0$ and $n \geq 3$, there exists a number $\delta = \delta(\kappa, n) < 1$ such that if a simply connected n-manifold M satisfies $K_M \geq -\kappa^2$ and has a point p with $\delta < K_p^{2\pi} \leq 1$, then M is diffeomorphic to S^n.*

If $n = 2$ then Theorems A and B are clear. Thus we will only prove them in the case $n \geq 3$.

It should be noted that if M admits the metric of a model (see [3]) $ds^2 = dr^2 + f(r)^2 d\Theta^2$, where $d\Theta^2$ is the canonical metric of S^{n-1} and $f\colon [0, d] \to \mathbf{R}^+$ satisfies $f > 0$ on $(0, d)$, $f(0) = f(d) = 0$, $f'(0) = -f'(d) = 1$, and $f'' < 0$, then $0 < a \leq K_p \leq b$ implies $a \leq K_M \leq b$. The same property holds for homogeneous Riemannian manifolds, however, it is not certain in general whether or not the positive radial curvature condition implies the same pinching for K_M.

The following example shows that for an arbitrary metric, pinched radial curvature does not necessarily impose pinching on sectional curvature.

Example. We consider a metric $ds^2 = dr^2 + f(r)^2(d\theta_1^2 + (\sin^2\theta_1)\, d\theta_2^2)$ on $\mathbf{R}^3$, where (r, θ_1, θ_2) is the polar coordinate about $p := (0, 0, 0)$ and f is a smooth function on $[0, \infty)$ defined as follows:

$$f(r) > 0 \quad \text{on } (0, \infty), \qquad f(r) = \begin{cases} \sin r & \text{on } [0, \pi/2], \\ e^{-r} & \text{on } [\pi, \infty). \end{cases}$$

Then the radial curvature from p takes values on an interval $[a, b]$, but $\lim_{r\to\infty} K_{ds^2}(\partial/\partial\theta_1 \wedge \partial/\partial\theta_2) = \infty$.

1. Preparation

In this section we will observe the behavior of radial curvature.

Put $d(p) := \sup_{q\in M} d(p, q)$ for $p \in M$, where $d(*, *)$ is the distance function on M. It is clear by Myers' theorem that if there exist a point $p \in M$ and a positive number δ such that $\delta \leq K_p^{\min}$, then $d(p) \leq \pi\sqrt{\delta}$, and consequently M is compact.

Next we shall prove Toponogov's comparison theorem for the radial curvature. M^δ denotes the complete simply connected surface of constant curvature δ.

Proposition 1.1. *Let p be a point of M with $K_p^{\min} \geq \delta$. Let $\Delta(\gamma_1, \gamma_2, \gamma_3)$ be a minimal geodesic triangle in M such that $\gamma_2(0) = \gamma_1(l_1) = p$, $\gamma_2(l_2) = \gamma_3(0)$, and $\gamma_3(l_3) = \gamma_1(0)$, where $l_i = L(\gamma_i)$ for $i = 1, 2, 3$. Then there exists a minimal geodesic triangle $\Delta(\tilde\gamma_1, \tilde\gamma_2, \tilde\gamma_3)$ in M^δ such that*

$$L(\gamma_i) = L(\tilde\gamma_i), \tag{1.1}$$

$$\sphericalangle(\dot\gamma_1(0), -\dot\gamma_3(l_3)) \geq \sphericalangle(\dot{\tilde\gamma}_1(0), -\dot{\tilde\gamma}_3(l_3)), \tag{1.2}$$

and

$$\sphericalangle(-\dot\gamma_2(l_2), \dot\gamma_3(0)) \geq \sphericalangle(-\dot{\tilde\gamma}_2(l_2), \dot{\tilde\gamma}_3(0)). \tag{1.3}$$

In particular, if $\delta > 0$ then for arbitrary $q, r \in M$, $d(p, q)+d(p, r)+d(q, r) \leq 2\pi/\sqrt{\delta}$ and $d(q, r) \leq \pi/\sqrt{\delta}$ (i.e., the diameter of M does not exceed $\pi/\sqrt{\delta}$).

Proof. We may assume that

$$\sphericalangle(\dot\gamma_1(0), -\dot\gamma_3(l_3)) \neq 0 \quad \text{and} \quad \sphericalangle(-\dot\gamma_2(l_2), \dot\gamma_3(0)) \neq 0$$

because the proposition is clear when one of them equals zero.

To conclude the proof it suffices to work with $M^{\delta-\varepsilon}$ instead of M^δ, where ε is an arbitrary small positive number. We only prove the case $\delta > 0$; if $\delta \leq 0$, the proof is easier. The proof is divided into three steps. Let τ_s be a minimal geodesic from $\gamma_3(s)$ to p and let L_s denote $L(\tau_s)$.

Step 1. If s is sufficiently small, then Proposition 1.1 holds for the triangle $\Delta(\tau_s, \gamma_2, \gamma_3|[0, s])$.

Proof. Since $L_s \leq \pi/\sqrt{\delta}$ and $l_2 \leq \pi/\sqrt{\delta}$, it follows that $L_s+l_2+s < 2\pi/\sqrt{\delta-\varepsilon}$ for small s. Therefore $\Delta(\tilde\tau_s, \tilde\gamma_2, \tilde\gamma_s)$ satisfying (1.1) for $\Delta(\tau_s, \gamma_2, \gamma_3|[0, s])$ is uniquely determined in $M^{\delta-\varepsilon}$. It suffices to show that $\Delta(\tilde\tau_s, \tilde\gamma_2, \tilde\gamma_s)$ satisfies (1.3). We extend γ_2 to the map on $(-\varepsilon_1, l_2 + \varepsilon_1)$ for small positive number ε_1. Let $P(t)$ be the parallel vector field along γ_2 such that $P(l_2) = \dot\gamma_3(0)$. We consider a map $S\colon (-\varepsilon_1, l_2 + \varepsilon_1) \times (-s, s) \to M$ such that $S(t, u) := \exp_{\gamma(t)} uP(t)$. We may assume $\sphericalangle(-\dot\gamma_2(l_2), \dot\gamma_3(0)) \neq \pi$. For sufficiently small s, S is a 2-dimensional submanifold of M. By a direct calculation,

$$G(t, 0) = K_M(\dot\gamma_2(t) \wedge P(t)),$$

where G is the Gauss curvature of S relative to the induced metric. Thus we may assume $G \geq \delta - \varepsilon$ over S by replacing s with a smaller number if

necessary. Let $\hat{\gamma}$ be a minimal geodesic in $M^{\delta-\varepsilon}$ such that $\hat{\gamma}(0) = \tilde{\gamma}_2(l_2)$ and $\sphericalangle(-\dot{\tilde{\gamma}}_2(l_2), \dot{\hat{\gamma}}(0)) = \sphericalangle(-\dot{\gamma}_2(l_2), \dot{\gamma}_3(0))$. Then Berger's comparison theorem (see [1, Theorem 1.29]) implies $d_M(\gamma_2(0), \gamma_3(s)) \leq d_{M^{\delta-\varepsilon}}(\tilde{\gamma}_2(0), \hat{\gamma}(s))$, and consequently $\Delta(\tilde{\tau}_s, \tilde{\gamma}_2, \tilde{\gamma}_s)$ satisfies (1.3).

Step 2. If there exists a unique minimal geodesic triangle $\Delta(\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3)$ in $M^{\delta-\varepsilon}$ that satisfies (1.1), then it also satisfies (1.2) and (1.3).

Proof. By Step 1 and the compactness of the image of γ_3 we can choose a subdivision $\{s_i\}_{i=1,2,\dots,k}$ of $[0, l_3]$ such that the proposition holds for the triangle $\Delta(\tau_{s_{i+1}}, \tau'_{s_i}, \gamma_3|[s_i, s_{i+1}])$, where τ'_s is the reversed geodesic for τ_s. We can prove Step 2 inductively by the convexity character of $M^{\delta-\varepsilon}$.

Step 3. For any minimal geodesic triangle $\Delta(\gamma_1, \gamma_2, \gamma_3)$ in M, there exists a unique minimal geodesic triangle $\Delta(\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3)$ in $M^{\delta-\varepsilon}$ that satisfies (1.1).

Proof. We put $s_0 := \sup\{s_1 \in [0, l_3] |$ there exists for any $s < s_1$ a unique minimal geodesic triangle $\Delta(\tilde{\tau}_s, \tilde{\gamma}_2, \tilde{\gamma}_s)$ in $M^{\delta-\varepsilon}$ such that $L(\tilde{\tau}_s) = L(\tau_s)$ and $L(\tilde{\gamma}_s) = s\}$. Then Step 1 implies that $s_0 > 0$. Suppose that $s_0 \neq l_3$. It is clear that $L(\tau_{s_0}) + l_2 + s_0 = 2\pi/\sqrt{\delta - \varepsilon}$ and $s_0 \leq \pi/\sqrt{\delta - \varepsilon}$. If $s_0 = \pi/\sqrt{\delta - \varepsilon}$ then $\tau_{s_0} \cup \gamma_2$ is a minimal geodesic through p of length $\pi/\sqrt{\delta - \varepsilon}$ that is greater than $\pi/\sqrt{\delta}$. This contradicts the fact that $\tau_{s_0} \cup \gamma_2$ contains a conjugate pair. If $s_0 < \pi/\sqrt{\delta - \varepsilon}$ then, by Step 2, $\sphericalangle(-\dot{\gamma}_2(l_2), \dot{\gamma}_3(0)) = \sphericalangle(-\dot{\gamma}_3(l_3), \dot{\tau}_{s_0}(0)) = \pi$. Therefore $\tau_{s_0} = \gamma_3|[s_0, l_3] \cup \gamma_1$. This contradicts the definition of s_0.

The proof of Proposition 1.1 is complete. □

For $p \in M$, a point x of M is called a critical point of the distance function $d(p, *)$ from p if for any $\nu \in S_x M$ there is a minimal geodesic γ joining x to p such that $\langle \nu, \dot{\gamma}(0) \rangle \geq 0$, where $\langle\ ,\ \rangle$ is the Riemannian metric. In particular, p and the farthest points from p are critical points of $d(p, *)$.

Corollary 1.1. *Let p be a point of M with $K_p^{\min} \geq \delta > 0$. If $d(p) > \pi/2\sqrt{\delta}$ then there exists a unique point $q \in M$ that satisfies $d(p, q) = d(p)$. Moreover, $d(p, *)$ and $d(q, *)$ have no critical points except q in $M - B_p(\pi/2\sqrt{\delta})$.*

Proof. The existence of q is clear. Suppose that there is another point r that is a critical point of $d(p, *)$ in $M - B_p(\pi/2\sqrt{\delta})$. Let σ be a minimal geodesic with $\sigma(0) = q$ and $\sigma(l) = r$. Then there exist minimal geodesics γ_1 joining q to p and γ_2 joining r to p such that $\sphericalangle(\dot{\sigma}(0), \dot{\gamma}_1(0)) \leq \pi/2$ and $\sphericalangle(-\dot{\sigma}(l), \dot{\gamma}_2(0)) \leq \pi/2$. It is a contradiction to Proposition 1.1. Similarly there are no critical points of $d(q, *)$ except q in $M - B_p(\pi/2\sqrt{\delta})$. □

Corollary 1.2. *Let p be a point of M with $K_p^{\min} \geq 1$. If $d(p) = \pi$ then M is isometric to $S^n(1)$.*

Proof. Let q be the point with $d(p, q) = d(p)$. By Proposition 1.1 and the triangle inequality, all geodesics from p reach q at time π and, therefore, minimal to q. It follows that $K_p \equiv 1$ and M is isometric to $S^n(1)$ (see [1, Theorem 6.5]). □

2. Proof of Theorem A

To prove Theorem A we shall recall the following lemma shown by Klingenberg.

Lemma 2.1 [5, Lemma (1.3)]. *Under the assumption of Theorem* A, *if* $n \geq 3$ *then* $i(p) \geq \pi$, *where* $i(p)$ *is the injectivity radius at* p.

Corollary 2.1. *Under the assumption of Theorem* A, *if* $n \geq 3$ *then* $d(p) \geq \pi$. *Here equality holds if and only if* M *is isometric to* $S^n(1)$.

Proof of Corollary 2.1. If $d(p) = \pi$, then any geodesic emanating from p reaches the unique farthest point q from p at time π. This follows from Lemma 2.1 and Corollary 1.1. Therefore $K_p \equiv 1$ and M is isometric to S^n. □

Proof of Theorem A. Put $\delta := \min\{K_p^{2\pi}\}$. By Lemma 2.1, $B_p(\pi/2\sqrt{\delta})$ is diffeomorphic to the n-disc D^n since $\pi/2\sqrt{\delta} < \pi$. Furthermore, $\partial B_p(\pi/2\sqrt{\delta})$ is diffeomorphic to S^{n-1}. Thus by Corollary 1.1, $M - B_p(\pi/2\sqrt{\delta})$ is diffeomorphic to D^n (see [1]). In particular, M is a twisted sphere and consequently homeomorphic to S^n. □

3. Proof of Theorem B

First we construct Hausdorff approximation maps between M and $S^n \subset \mathbf{R}^{n+1}$ under the assumption of Theorem B.

By Lemma 2.1, if $\delta \geq \frac{1}{4}$ then $i(p) \geq \pi$. We consider two maps f and f' from $B_p(\pi)$ to $S^n(1)$. For $\tilde{p} \in S^n(1)$ and $\hat{p} \in S^n(\delta)$, let $I_1 \colon T_pM \to T_{\tilde{p}}S^n(1)$ and $I_\delta \colon T_pM \to T_{\hat{p}}S^n(\delta)$ be linear isometries, where $S^n(\delta)$ is the n-sphere of constant curvature δ. Let g be the radial projection from $S^n(\delta)$ to $S^n(1)$ in $\mathbf{R}^{n+1}$. We put

$$h := \exp_{\hat{p}} \circ I_\delta \circ (\exp_p |B_p(\pi))^{-1} \colon B_p(\pi) \to S^n(\delta),$$
$$f' := g \circ h \colon B_p(\pi) \to S^n,$$

and

$$f := \exp_{\tilde{p}} \circ I_1 \circ (\exp_p |B_p(\pi))^{-1} \colon B_p(\pi) \to S^n.$$

Then it follows from Rauch's comparison Theorem and Proposition 2.1 that

$$d(x, y) \leq d_{S(\delta)}(h(x), h(y)) + 2\pi(1/\sqrt{\delta} - 1)$$

and

$$d_{S(1)}(f(x), f(y)) \leq d(x, y)$$

for $x, y \in B_p(\pi)$, where $d_{S(\delta)}$ (resp. $d_{S(1)}$) is the distance function on $S^n(\delta)$ (resp. $S^n(1)$). It is clear that g and g^{-1} give

$$d_{\mathrm{H}}(S^n(\delta), S^n(1)) \leq \pi(1/\sqrt{\delta} - 1)$$

and that

$$|d_S(f(x), f(y)) - d_S(f'(x), f'(y))| \leq 2\pi(1/\sqrt{\delta} - 1) \quad \text{for } x, y \in B_p(\pi),$$

where d_{H} is the Hausdorff distance.

Thus we obtain

Lemma 3.1. *There exists a positive number* $\varepsilon = \varepsilon(\delta)$ *such that* $\lim_{\delta \to 1} \varepsilon = 0$ *and* $f \colon B_p(\pi) \to S^n$ *is an* ε*-Hausdorff approximation map.*

Extend f and f^{-1} to the maps on M and $S^n(1)$ by $f|M - B_p(\pi) := -\tilde{p}$ and $f^{-1}(-\tilde{p}) :=$ the farthest point from p, where $-\tilde{p}$ is the antipodal point of $\tilde{p}$. Since these extended $f \colon M \to S^n(1)$ and $f^{-1} \colon S^n(1) \to M$ are also ε-Hausdorff approximation maps, we have

Proposition 3.1. *For $\delta < 1$ close to 1 there exists a positive number $\varepsilon = \varepsilon(\delta)$ with $\lim_{\delta \to 1} \varepsilon = 0$ such that if a simply-connected manifold M has a point p with $\delta < K_p^{2\pi} \leq 1$, then $d_{\mathrm{H}}(M, S^n(1)) < \varepsilon$.*

The following proposition is an immediate consequence of Yamaguchi's fibration theorem.

Proposition 3.2. *For given $\kappa \geq 0$ and $n \geq 2$, there exists a positive number $\varepsilon = \varepsilon(\kappa, n)$ such that if an n-manifold M with $K_M \geq -\kappa^2$ satisfies $d_{\mathrm{H}}(M, S^n(1)) < \varepsilon$, then M is diffeomorphic to $S^n(1)$.*

The proof of Theorem B is now complete.

Remark. Since the target manifold is $S^n(1)$, we can obtain a better estimate for ε than Yamaguchi's by the same way as the proof of the following theorem.

Theorem (Otsu, Shiohama, and Yamaguchi [8]). *There exists a positive number $\varepsilon(n)$ such that if $K_M \geq 1$ and $\mathrm{Vol}(M) \geq \omega_n - \varepsilon(n)$, then M is diffeomorphic to S^n, where $\mathrm{Vol}(M)$ is the volume of M and ω_n is the volume of the canonical unit sphere $S^n(1)$.*

The basic idea of proving the above theorem is to construct an embedding Φ of M into $\mathbf{R}^{n+1}$. The regularity of Φ is obtained by Toponogov's theorem and Hausdorff closeness between M and S^n. A crucial point of the proof of the regularity of Φ in [8] is the angle estimate of every geodesic triangle sitting in a general position in M, in which Toponogov's theorem plays an important role. However, in our case we cannot develop the same discussion because we cannot compare the angle of every geodesic triangle in a general position whose vertices do not contain the base point p. Nevertheless, the desired angle estimate is obtained from the Hausdorff closeness between M and S^n by assuming the lower bound for K_M. For the case $\kappa = 0$, we obtain the estimate $\varepsilon = 10^{-4}(n+1)^{-4}(2(n+1)^{3/2} + (n+1)^{1/2} + 4)^{-2}(4n^2 - 3)^{-2}$.

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DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, KYUSHU UNIVERSITY, FUKUOKA 812, JAPAN