PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 118, Number 3, July 1993

THE BANACH-MAZUR GAME AND GENERIC EXISTENCE OF SOLUTIONS TO OPTIMIZATION PROBLEMS

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(Communicated by Franklin D. Tall)

ABSTRACT. The existence of a winning strategy in the well-known Banach-Mazur game in a completely regular topological space X is proved to be equivalent to the generic existence of solutions of optimization problems generated by continuous functions in X.

1. INTRODUCTION

Let X be a completely regular topological space and $C(X)$ denote the space of all continuous and bounded real-valued functions in X. Equipping $C(X)$ with the usual sup-norm $\|f\| := \sup\{|f(x)|: x \in X\}$, under which $C(X)$ is a Banach space, the following question makes sense: Under what conditions (necessary and sufficient) on X does the set $E := \{f \in C(X): f$ attains its minimum in $X\}$ contain a dense and G_δ-subset of $C(X)$? That is, under what assumptions on X is the set E residual in $C(X)$? Call this property "generic existence of solutions" of the minimization problems generated by the functions from $C(X)$.

It turns out that the generic existence of the solutions to the minimization problems for functions from $C(X)$ is related to the following topological game in X. Two players, named α and β, play a game in X in the following way: β chooses first a nonempty open subset U_1 of X. Then α chooses a nonempty open subset V_1 with $V_1 \subset U_1$. Further, β chooses a nonempty open subset U_2 of X with $U_2 \subset V_1$ and α chooses a nonempty open $V_2 \subset U_2$ and so on. The so-obtained infinite sequence $U_1, V_1, \dots$ is called a play. The player α wins this play if $\bigcap_{n=1}^{\infty} V_n \neq \varnothing$. Otherwise β wins.

This game is one of the most known modifications of the Banach-Mazur game and is denoted usually by $BM(X)$. For the terminology and facts that we use the reader is referred to the survey [Tel]. Under a strategy for the player α in the game $BM(X)$ we understand a mapping s that assigns to every chain $(U_1, V_1, \dots, U_n)$ corresponding to the first legal n moves of β and the first $n-1$ moves of α, $n \geq 1$, a nonempty open set $V_n \subset U_n$. The strategy s is called winning strategy for the player α (or α-winning strategy) if for every

Received by the editors November 20, 1991.
1991 *Mathematics Subject Classification*. Primary 90D44, 90C48.
Key words and phrases. Banach-Mazur game, winning strategy, generic, optimization problems.

infinite sequence of open sets $U_1 \supset V_1 \supset \cdots \supset U_n \supset V_n \supset \cdots$ such that $V_n = s(U_1, V_1, \dots, U_n)$ for every $n \geq 1$, we have $\bigcap_{n=1}^{\infty} V_n \neq \varnothing$. A stationary winning strategy (called also α-winning tactic (see [Ch])) for the player α in the game $BM(X)$ is a strategy for α that on each step depends only on the last move of the player β. Precisely, a stationary winning strategy t for the player α is a mapping from the family of nonempty open subsets of X into the family of nonempty open subsets of X such that for every nonempty open $U \subset X$ one has $t(U) \subset U$ and, moreover, whenever one has a sequence $(U_n)_{n\geq 1}$ such that $U_{n+1} \subset t(U_n)$ for every n, then $\bigcap_{n=1}^{\infty} U_n \neq \varnothing$. Evidently, every α-winning tactic t determines the winning strategy $s(U_1, V_1, \dots, U_n) := t(U_n)$. There are, however, spaces X (see [De]) with a winning strategy for the player α that do not admit an α-winning tactic. Every space X that admits a winning strategy for the player α in the game $BM(X)$ is a Baire space.

It was proved in [St] that if X possesses an α-winning tactic, then one has generic existence of the solution to the minimization problems generated by the functions from $C(X)$. We show here that this result can be obtained under weaker assumptions on X. It suffices to suppose that X admits only a winning strategy for the player α in the game $BM(X)$ in order to have generic existence of solutions. Moreover, Theorem 3.1 asserts that the generic existence of solutions of the minimization problems determined by functions from $C(X)$ is a characterization of the fact that X admits a winning strategy for the player α in $BM(X)$. As a corollary we give also another characterization of the spaces X that possesses an α-winning strategy (α-winning tactic) in the case when X has a σ-discrete net.

2. Some preliminaries

Throughout this article only completely regular topological spaces will be considered. For a subset A of the topological space X we denote by $\mathrm{Int}_X(A)$ and $\mathrm{Cl}_X(A)$ the interior and the closure of A in X. If there is no danger of confusion we will write simply $\mathrm{Int}(A)$ and $\mathrm{Cl}(A)$ correspondingly.

For a function $f \in C(X)$ denote by $M(f)$ the set (possibly empty) of the minimizers of f in X; i.e.,

$$M(f) := \{x \in X : f(x) = \inf\{f(y) : y \in X\} =: \inf(X, f)\}.$$

Hence M is a multivalued mapping from $C(X)$ onto X that may have empty values. It can be seen that the domain of M, that is the set $\mathrm{Dom}(M) := \{f \in C(X) : M(f) \neq \varnothing\}$, is dense in $C(X)$. Indeed, if $f \in C(X)$ and $\varepsilon > 0$ are arbitrary, then obviously $M(f_\varepsilon) \neq \varnothing$, where $f_\varepsilon(x) = \max\{f(x), \inf(X, f) + \varepsilon\}$. However, the set $\mathrm{Dom}(M)$ is not obliged to contain a dense and G_δ-subset of $C(X)$.

For a subset U of X put $M^{\#}(U) := \{f \in C(X) : M(f) \subset U\}$ and for $W \subset C(X)$ let $M(W) := \bigcup\{M(f) : f \in W\}$. Further, given arbitrary $f \in C(X)$ and $\varepsilon > 0$, denote by $\Omega_f(\varepsilon)$ the set $\{x \in X : f(x) < \inf(X, f) + \varepsilon\}$. The sets $\Omega_f(\varepsilon)$ are nonempty and open for every $\varepsilon > 0$. Obviously $\Omega_f(\varepsilon_1) \subset \Omega_f(\varepsilon_2)$ provided $\varepsilon_1 \leq \varepsilon_2$ and, moreover, $M(f) = \bigcap_{\varepsilon>0} \Omega_f(\varepsilon)$.

The following proposition summarizes some properties of the mapping M that we will use later.

Proposition 2.1. *The mapping M has the following properties:*

(a) *M is open, i.e., $M(W)$ is (nonempty) and open in X provided W is (nonempty) and open in $C(X)$;*

(b) $\operatorname{Int} M^{\#}(U) \neq \varnothing$ *for every nonempty open U in X;*

(c) *for every two open sets $W \subset C(X)$ and $U \subset X$, respectively, with $M(W) \cap U \neq \varnothing$ there exists a nonempty open $W' \subset W$ such that $M(W') \subset U$;*

(d) *let $\{f_0\} = \bigcap_{n=1}^{\infty} B_n$, where $(B_n)_{n\geq 1}$ is a decreasing sequence of subsets in $C(X)$ with $\lim \operatorname{diam}(B_n) = 0$. Then $M(f_0) = \bigcap_{n=1}^{\infty} M(B_n)$.*

Proof. (a) Let W be an open subset of $C(X)$ and $x_0 \in M(f_0)$ for some $f_0 \in W$. Take $\varepsilon > 0$ such that the ball $B(f_0, \varepsilon) := \{f \in C(X) \colon \|f - f_0\| < \varepsilon\} \subset W$. Then each $x \in \Omega_{f_0}(\varepsilon)$ is a minimizer of some f from W, e.g., of the function f_ε considered above.

(b) Let $x_0 \in U$. Since X is completely regular, there exists $h_0 \in C(X)$ with $h_0(x_0) = 0$, $h_0(X \setminus U) = 1$, and $\|h\| \leq 1$. It is easy to see that $M(B(h_0, 1/3)) \subset U$.

(c) Let $x_0 \in M(f_0) \bigcap U$ for some $f_0 \in W$. Consider the function h_0 from (b) and find $\delta > 0$ such that $f_0 + \delta h_0 \in W$. Let further, $W' \subset W$ be an open set in $C(X)$ containing $f_0 + \delta h_0$ and such that $\operatorname{diam}(W') < \delta/3$. Take $f \in W'$. Since for $x \in X \setminus U$ one has $f(x) \geq (f_0 + \delta h_0)(x) - \delta/3 = f_0(x) + (2\delta)/3 \geq f_0(x_0) + (2\delta)/3 = (f_0 + \delta h_0)(x_0) + (2\delta)/3 > (f_0 + \delta h_0)(x_0) + \delta/3 \geq f(x_0)$, we see that $M(f) \subset U$.

(d) Obviously $M(f_0) \subset \bigcap_{n=1}^{\infty} M(B_n)$. On the other hand, take some $x \in M(B_n)$. Then $x \in M(f_n)$ for some $f_n \in B_n$ with $\|f_n - f_0\| \leq \operatorname{diam}(B_n)$. Hence, $x \in \Omega_{f_0}(2\operatorname{diam}(B_n))$. Therefore, $M(B_n) \subset \Omega_{f_0}(2\operatorname{diam}(B_n))$. This gives $\bigcap_{n=1}^{\infty} M(B_n) \subset \bigcap_{n=1}^{\infty} \Omega_{f_0}(2\operatorname{diam}(B_n)) = M(f_0)$. □

3. The main result

The main result in this article is the following characterization of the fact that the space X possesses an α-winning strategy.

Theorem 3.1. *The space X admits a winning strategy for the player α in the Banach-Mazur game $BM(X)$ if and only if* $\operatorname{Dom}(M)$ *contains a dense and G_δ-subset of $C(X)$.*

Proof. Sufficiency. Suppose $\operatorname{Dom}(M)$ contains a dense and G_δ-subset of $C(X)$. Then there exist countably many open and dense subsets $(G_n)_{n\geq 1}$ of $C(X)$ such that $\bigcap_{n=1}^{\infty} G_n \subset \operatorname{Dom}(M)$. The sets $F_n := C(X) \setminus G_n$, $n \geq 1$, are closed and nowhere dense in $C(X)$. That is $\operatorname{Int}(F_n) = \varnothing$ for every $n \geq 1$.

We show that the player α has a winning strategy s in the game $BM(X)$. Let U_1 be a nonempty open subset of X. Consider the set $\operatorname{Int} M^{\#}(U_1)$ that is nonempty by Proposition 2.1(b). Since F_1 is closed and nowhere dense in $C(X)$, the set $\operatorname{Int} M^{\#}(U_1) \setminus F_1$ is nonempty and open in $C(X)$. Take an open ball B_1 in $C(X)$ with radius less or equal to 1, such that $B_1 \subset \operatorname{Int} M^{\#}(U_1) \setminus F_1$. Define now the value of the strategy s at U_1 by $s(U_1) := M(B_1)$. By Proposition 2.1(a), $s(U_1)$ is a nonempty open subset of U_1.

Further, let U_2 be an arbitrary nonempty open subset of $V_1 = s(U_1) = M(B_1)$. Since $U_2 \subset M(B_1)$ there is some $f \in B_1$ such that $M(f) \cap U_2 \neq \varnothing$. Hence, by Proposition 2.1(c) there exists a nonempty open $W \subset B_1$ such

that $M(W) \subset U_2$. As above the set $W \setminus F_2$ is a nonempty and open subset of $C(X)$. Take an open ball B_2 with radius less or equal to $1/2$ such that $\mathrm{Cl}(B_2) \subset W \setminus F_2 \subset B_1$ and put $s(U_1, V_1, U_2) := M(B_2)$. Obviously $s(U_1, V_1, U_2)$ is a nonempty open subset of U_2. Proceeding by induction we define s for every chain $(U_1, V_1, \dots, U_n)$, $n \geq 1$, such that $U_k \subset V_{k-1}$ and $V_{k-1} = s(U_1, V_1, \dots, U_{k-1})$ for every k, $2 \leq k \leq n$.

Let $U_1 \supset V_1 \supset \cdots \supset U_n \supset V_n \supset \cdots$ be an infinite sequence of open sets in X such that for every $n \geq 1$, $V_n = s(U_1, V_1, \dots, U_n)$. Let $(B_n)_{n\geq 1}$ be the sequence of open balls in $C(X)$ associated with $(U_n)_{n\geq 1}$ and $(V_n)_{n\geq 1}$ from the construction of s. Then for every $n \geq 1$:

(1) $\mathrm{Cl}(B_{n+1}) \subset B_n$ and $B_n \bigcap F_n = \varnothing$;

(2) $\mathrm{diam}(B_n) < 1/n$;

(3) $V_n = M(B_n)$.

Conditions (1) and (2) guarantee that $\bigcap_{n=1}^{\infty} B_n$ is a one-point set in $C(X)$, say f_0. Moreover, (1) shows in addition that $f_0 \in C(X) \setminus \bigcup_{n=1}^{\infty} F_n \subset \mathrm{Dom}(M)$. Therefore, by (3) and Proposition 2.1(d) we have

$$\varnothing \neq M(f_0) = \bigcap_{n=1}^{\infty} M(B_n) = \bigcap_{n=1}^{\infty} V_n.$$

Hence s is a winning strategy for the player α in the Banach-Mazur game $BM(X)$.

Necessity. Suppose now, that there exists a winning strategy s for the player α in the game $BM(X)$. Every finite sequence of sets $(U_1, V_1, \dots, U_n, V_n)$, $n \geq 1$, obtained by the first n steps in the game $BM(X)$ is called a partial play in this game. The key step in the proof is the following lemma:

Lemma 3.2. *Let $(U_1, V_1, \dots, U_n, V_n)$, $n \geq 1$, be a partial play in the game $BM(X)$ and W_n be a nonempty open subset of $C(X)$ such that $M(W_n) \subset V_n$. Then there is a family $\Gamma(W_n)$ of triples $(U_{n+1}, V_{n+1}, W_{n+1})$ such that:*

(a) *U_{n+1} is a nonempty open subset of $M(W_n)$;*

(b) *$V_{n+1} = s(U_1, V_1, \dots, U_n, V_n, U_{n+1})$;*

(c) *W_{n+1} is a nonempty subset of $C(X)$ such that $\mathrm{diam}(W_{n+1}) < 1/(n+1)$, $\mathrm{Cl}(W_{n+1}) \subset W_n$, and $M(W_{n+1}) \subset V_{n+1}$;*

(d) *the family $\gamma(W_n) := \{W_{n+1} : (U_{n+1}, V_{n+1}, W_{n+1}) \in \Gamma(W_n)$ for some U_{n+1}, $V_{n+1}\}$ is disjoint;*

(e) *the set $H(W_n) := \bigcup\{W_{n+1} : W_{n+1} \in \gamma(W_n)\}$ is dense in W_n.*

Proof of Lemma 3.2. Take a maximal family $\Gamma(W_n)$ satisfying the properties (a)–(d). We prove that it satisfies also the condition (e).

Suppose the contrary. There exists a nonempty open subset G of $C(X)$ with $G \subset W_n$ and $G \cap H(W_n) = \varnothing$. Consider the set $M(G)$ that, by Proposition 2.1(a), is a nonempty and open subset of X. Moreover, $M(G) \subset M(W_n) \subset V_n$. Let $U_{n+1} := M(G)$ and $V_{n+1} := s(U_1, V_1, \dots, U_n, V_n, U_{n+1})$. By Proposition 2.1(c) there is a nonempty open subset W_{n+1} of $C(X)$ such that $W_{n+1} \subset G$ and $M(W_{n+1}) \subset V_{n+1}$. We may arrange, in addition, $\mathrm{Cl}(W_{n+1}) \subset W_n$ and $\mathrm{diam}(W_{n+1}) < 1/(n+1)$. Now, the family $\Gamma' := \Gamma(W_n) \cup \{(U_{n+1}, V_{n+1}, W_{n+1})\}$ is strictly larger than $\Gamma(W_n)$ and satisfies (a)–(d). This is a contradiction showing that the maximal family $\Gamma(W_n)$ satisfies also (e). $\square$

Let us mention that Lemma 3.2 is true also for $n = 0$ provided we put $U_0 = V_0 = X$. Now, let us get back to the proof of the theorem. We proceed in the following way.

Put $\gamma_0 := \{C(X)\}$, $W_0 = C(X)$, $U_0 = V_0 = X$ and apply Lemma 3.2 for the triple (U_0, V_0, W_0). We get a family of triples $\Gamma_1 := \Gamma_1(W_0)$ satisfying conditions (a)–(e) from Lemma 3.2. Put $\gamma_1 := \gamma(W_0)$ and $H_1 := H(W_0)$. By (e) the set H_1 is open and dense in $C(X)$. Further, because of (d), for every $W_1 \in \gamma_1$ there is a unique couple (U_1, V_1) with $(U_1, V_1, W_1) \in \Gamma_1$. Apply again Lemma 3.2 for this triple. As a result, for every $W_1 \in \gamma_1$ we obtain a family of triples $\Gamma(W_1)$ with the properties (a)–(e) fulfilled with respect to the couple (U_1, V_1) corresponding to W_1. Let $\Gamma_2 := \bigcup\{\Gamma(W_1)\colon W_1 \in \gamma_1\}$, $\gamma_2 := \bigcup\{\gamma(W_1)\colon W_1 \in \gamma_1\}$, and $H_2 := \bigcup\{H(W_1)\colon W_1 \in \gamma_1\}$. Since γ_1 is disjoint and each $\gamma(W_1)$ is disjoint too, then the family γ_2 is also disjoint. Moreover, by (e) every $H(W_1)$ is dense in W_1 and since H_1 is dense in $C(X)$, it follows that H_2 is (open) and dense in $C(X)$ as well.

Proceeding in this way we obtain a sequence of families $(\Gamma_n)_{n\geq 1}$ of triples and a sequence of disjoint families $(\gamma_n)_{n\geq 0}$ of open sets in $C(X)$, with $\gamma_0 = \{C(X)\}$, such that for every $n \geq 1$ we have

(i) Γ_n is a union of the families $\Gamma(W_{n-1})$, $W_{n-1} \in \gamma_{n-1}$, where $\Gamma(W_{n-1})$ is obtained by Lemma 3.2 from some uniquely determined partial play $(U_1, V_1, \dots, U_{n-1}, V_{n-1})$;

(ii) γ_n is a union of the families $\gamma(W_{n-1})$, $W_{n-1} \in \gamma_{n-1}$ from the condition (d) of Lemma 3.2;

(iii) the set $H_n := \bigcup\{W_n\colon W_n \in \gamma_n\}$ is open and dense in $C(X)$.

Let $H_0 := \bigcap_{n=1}^{\infty} H_n$. Obviously H_0 is a dense and G_δ-subset of $C(X)$. Take $f_0 \in H_0$. By the properties above, this f_0 determines a unique sequence $(W_n)_{n\geq 1}$ such that for every $n \geq 1$, $W_n \in \gamma_n$, $\mathrm{Cl}(W_{n+1}) \subset W_n$, and $\mathrm{diam}(W_n) < 1/n$. Hence $\{f_0\} = \bigcap_{n=1}^{\infty} W_n$. By the properties (a)–(d) from Lemma 3.2 and conditions (i)–(iii) above it follows that there is an infinite sequence of open sets

$$U_1 \supset V_1 \supset \cdots \supset U_n \supset V_n \supset \cdots$$

such that $V_n = s(U_1, V_1, \dots, U_n)$ and $U_{n+1} \subset M(W_n) \subset V_n$ for every $n \geq 1$.

Hence, by Proposition 2.1(d) we have

$$M(f_0) = \bigcap_{n=1}^{\infty} M(W_n) = \bigcap_{n=1}^{\infty} V_n = \bigcap_{n=1}^{\infty} U_n.$$

Since s is a winning strategy, we see that $M(f_0) = \bigcap_{n=1}^{\infty} V_n \neq \varnothing$. The proof is complete. □

As an immediate corollary from Theorem 3.1 we get the following sufficient condition for the set $\mathrm{Dom}(M)$ to be residual in $C(X)$.

Corollary 3.3 (see [St, Theorem 5]). *Let X admit an α-winning tactic in the Banach-Mazur game. Then the set* $\mathrm{Dom}(M)$ *contains a dense and G_δ-subset of* $C(X)$.

As mentioned above, there exists a completely regular space X with α-winning strategy that does not admit an α-winning tactic (see [De]).

A slight change in the proof of Theorem 3.1 gives us the possibility to characterize the spaces X for which the set $\{f \in C(X)\colon f$ attains its minimum in X at exactly one point$\}$ contains a dense and G_δ-subset of $C(X)$, i.e., to characterize the spaces X in which we have generic uniqueness of the solution of the minimization problems generated by functions from $C(X)$.

Theorem 3.4. *The set $\{f \in C(X)\colon f$ attains its minimum in X at exactly one point$\}$ contains a dense and G_δ-subset of $C(X)$ if and only if the space X admits an α-winning strategy s such that, whenever one has a sequence of open sets $U_1 \supset V_1 \cdots \supset U_n \supset V_n \supset \cdots$ with $V_n = s(U_1, V_1, \dots, U_n)$ for every $n \geq 1$, then $\bigcap_{n=1}^{\infty} V_n$ is a one-point set.*

Let us mention a class of spaces for which we have generic uniqueness of the solution for the minimization problems for the functions from $C(X)$. The topological space X is called fragmentable (see [JaRo]) if there is a metric ρ on it such that for every $\varepsilon > 0$ and every nonempty subset Y of X there exists a nonempty relatively open subset A of Y with $\rho\text{-diam}(A) < \varepsilon$. Further information about fragmentable spaces can be found in [Na, Ri].

Corollary 3.5. *Let X be a fragmentable space that admits an α-winning strategy. Then the set $\{f \in C(X)\colon f$ attains its minimum in X at exactly one point$\}$ contains a dense and G_δ-subset of $C(X)$.*

Proof. Let ρ be the metric on X that fragments it. Let $U_1, V_1, \dots, U_n$ be the first n steps of the player β and the first $n-1$ steps of α in the game $BM(X)$, $n \geq 1$. Take a nonempty open set $U_n' \subset U_n$ such that $\rho\text{-diam}(U_n') < 1/n$, and define $s'(U_1, V_1, \dots, U_n) := s(U_1, V_1, \dots, U_n')$. It is easy to check that the so obtained strategy s' satisfies the requirements of Theorem 3.4. □

We give some further corollaries of Theorem 3.1. Before that let us recall some notions. A minimization problem generated by some $f \in C(X)$ (which we will denote by (X, f)) is said to be Tikhonov well posed (see [Ti]) if it has unique solution $x_0 \in X$ and, moreover, every minimizing sequence $(x_n)_{n\geq 1}$ (i.e., $f(x_n) \to \inf(X, f)$) converges to this unique solution. If (X, f) is Tikhonov well posed then every minimizing net (not only every minimizing sequence) converges to its unique solution (see, e.g., [ČKR]). Let

$$T := \{f \in C(X)\colon (X, f) \text{ is Tikhonov well posed}\}.$$

The following fact is proved for compact spaces X in [ČK1, ČK2] and for an arbitrary X in [ČKR, Theorem 3.5]: The set T contains a dense and G_δ-subset of $C(X)$ iff the space X contains a dense and completely metrizable subspace.

A family γ of subsets of X is called a net in X if for every $x \in X$ and every open $U \subset X$, with $x \in U$, there exists $H \in \gamma$ such that $x \in H \subset U$. The space X possesses a σ-discrete net if there are countably many discrete families $(\gamma_n)_{n\geq 1}$ in X such that $\gamma := \bigcup_{n=1}^{\infty} \gamma_n$ forms a net in X. Recall that a family of subsets in X is discrete if every point in X has a neighborhood that intersects at most one element of the family. Every metric space has a σ-discrete net. The following is again a result proved in [ČKR, Theorem 5.6]: Suppose that X possesses a σ-discrete net. Then the set $T \cup (C(X) \setminus \mathrm{Dom}(M))$ contains a dense and G_δ-subset of $C(X)$.

In the special case when X has a σ-discrete net we can give another characterization of the fact that the space X admits an α-winning strategy. For a metric space X this characterization follows by a result of [Ox] and for a space X with a base of countable order can be found in [Wh].

Theorem 3.6. *Let X possess a σ-discrete net. Then X admits an α-winning strategy in the game $BM(X)$ if and only if X contains a dense and completely metrizable subspace.*

Proof. The proof is an immediate consequence from Theorem 3.1 and the mentioned results from [ČKR]. □

Every space X that contains a dense and completely metrizable subspace possesses α-winning tactic (see, e.g., [Tel]). Then we have

Corollary 3.7. *Let X possess a σ-discrete net. Then X admits α-winning strategy in the game $BM(X)$ if and only if it admits α-winning tactic in this game.*

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