THE BANACH-MAZUR GAME AND GENERIC EXISTENCE OF SOLUTIONS TO OPTIMIZATION PROBLEMS

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Abstract. The existence of a winning strategy in the well-known Banach-Mazur game in a completely regular topological space $X$ is proved to be equivalent to the generic existence of solutions of optimization problems generated by continuous functions in $X$.

1. Introduction

Let $X$ be a completely regular topological space and $C(X)$ denote the space of all continuous and bounded real-valued functions in $X$. Equipping $C(X)$ with the usual sup-norm $\|f\| := \sup\{|f(x)| : x \in X\}$, under which $C(X)$ is a Banach space, the following question makes sense: Under what conditions (necessary and sufficient) on $X$ does the set $E := \{f \in C(X) : f \text{ attains its minimum in } X\}$ contain a dense and $G_\delta$-subset of $C(X)$? That is, under what assumptions on $X$ is the set $E$ residual in $C(X)$? Call this property "generic existence of solutions" of the minimization problems generated by the functions from $C(X)$.

It turns out that the generic existence of the solutions to the minimization problems for functions from $C(X)$ is related to the following topological game in $X$. Two players, named $\alpha$ and $\beta$, play a game in $X$ in the following way: $\beta$ chooses first a nonempty open subset $U_1$ of $X$. Then $\alpha$ chooses a nonempty open subset $V_1$ with $V_1 \subset U_1$. Further, $\beta$ chooses a nonempty open subset $V_2$ of $X$ with $V_2 \subset V_1$ and $\alpha$ chooses a nonempty open $V_2 \subset V_1$ and so on. The so-obtained infinite sequence $U_1, V_1, \ldots$ is called a play. The player $\alpha$ wins this play if $\bigcap_{n=1}^{\infty} V_n \neq \emptyset$. Otherwise $\beta$ wins.

This game is one of the most known modifications of the Banach-Mazur game and is denoted usually by $BM(X)$. For the terminology and facts that we use the reader is referred to the survey [Tel]. Under a strategy for the player $\alpha$ in the game $BM(X)$ we understand a mapping $s$ that assigns to every chain $(U_1, V_1, \ldots, U_n)$ corresponding to the first legal $n$ moves of $\beta$ and the first $n - 1$ moves of $\alpha$, $n \geq 1$, a nonempty open set $V_n \subset U_n$. The strategy $s$ is called winning strategy for the player $\alpha$ (or $\alpha$-winning strategy) if for every
infinite sequence of open sets $U_1 \supset V_1 \supset \cdots \supset U_n \supset V_n \supset \cdots$ such that $V_n = s(U_1, V_1, \ldots, U_n)$ for every $n \geq 1$, we have $\cap_{n=1}^{\infty} V_n \neq \emptyset$. A stationary winning strategy (called also $\alpha$-winning tactic (see [Ch])) for the player $\alpha$ in the game $BM(X)$ is a strategy for $\alpha$ that on each step depends only on the last move of the player $\beta$. Precisely, a stationary winning strategy $t$ for the player $\alpha$ is a mapping from the family of nonempty open subsets of $X$ into the family of nonempty open subsets of $X$ such that for every nonempty open $U \subset X$ one has $t(U) \subset U$ and, moreover, whenever one has a sequence $(U_n)_{n \geq 1}$ such that $U_{n+1} \subset t(U_n)$ for every $n$, then $\cap_{n=1}^{\infty} U_n \neq \emptyset$. Evidently, every $\alpha$-winning tactic $t$ determines the winning strategy $s(U_1, V_1, \ldots, U_n) := t(U_n)$. There are, however, spaces $X$ (see [De]) with a winning strategy for the player $\alpha$ that do not admit an $\alpha$-winning tactic. Every space $X$ that admits a winning strategy for the player $\alpha$ in the game $BM(X)$ is a Baire space.

It was proved in [St] that if $X$ possesses an $\alpha$-winning tactic, then one has generic existence of the solution to the minimization problems generated by the functions from $C(X)$. We show here that this result can be obtained under weaker assumptions on $X$. It suffices to suppose that $X$ admits only a winning strategy for the player $\alpha$ in the game $BM(X)$ in order to have generic existence of solutions. Moreover, Theorem 3.1 asserts that the generic existence of solutions of the minimization problems determined by functions from $C(X)$ is a characterization of the fact that $X$ admits a winning strategy for the player $\alpha$ in $BM(X)$. As a corollary we give also another characterization of the spaces $X$ that possesses an $\alpha$-winning strategy ($\alpha$-winning tactic) in the case when $X$ has a $\sigma$-discrete net.

2. Some preliminaries

Throughout this article only completely regular topological spaces will be considered. For a subset $A$ of the topological space $X$ we denote by $\text{Int}_X(A)$ and $\text{Cl}_X(A)$ the interior and the closure of $A$ in $X$. If there is no danger of confusion we will write simply $\text{Int}(A)$ and $\text{Cl}(A)$ correspondingly.

For a function $f \in C(X)$ denote by $M(f)$ the set (possibly empty) of the minimizers of $f$ in $X$; i.e.,

$$M(f) := \{x \in X: f(x) = \inf\{f(y): y \in X\} =: \inf(X, f)\}.$$

Hence $M$ is a multivalued mapping from $C(X)$ onto $X$ that may have empty values. It can be seen that the domain of $M$, that is the set $\text{Dom}(M) := \{f \in C(X): M(f) \neq \emptyset\}$, is dense in $C(X)$. Indeed, if $f \in C(X)$ and $\varepsilon > 0$ are arbitrary, then obviously $M(f_\varepsilon) \neq \emptyset$, where $f_\varepsilon(x) = \max\{f(x), \inf(X, f) + \varepsilon\}$. However, the set $\text{Dom}(M)$ is not obliged to contain a dense and $G_\delta$-subset of $C(X)$.

For a subset $U$ of $X$ put $M^*(U) := \{f \in C(X): M(f) \subset U\}$ and for $W \subset C(X)$ let $M(W) := \bigcup\{M(f): f \in W\}$. Further, given arbitrary $f \in C(X)$ and $\varepsilon > 0$, denote by $\Omega_f(\varepsilon)$ the set \( \{x \in X: f(x) < \inf(X, f) + \varepsilon\} \). The sets $\Omega_f(\varepsilon)$ are nonempty and open for every $\varepsilon > 0$. Obviously $\Omega_f(\varepsilon_1) \subset \Omega_f(\varepsilon_2)$ provided $\varepsilon_1 \leq \varepsilon_2$ and, moreover, $M(f) = \cap_{\varepsilon > 0} \Omega_f(\varepsilon)$.

The following proposition summarizes some properties of the mapping $M$ that we will use later.
Proposition 2.1. The mapping $M$ has the following properties:
(a) $M$ is open, i.e., $M(W)$ is (nonempty) and open in $X$ provided $W$ is (nonempty) and open in $C(X)$;
(b) Int$M^*(U)$ $\neq \emptyset$ for every nonempty open $U$ in $X$;
(c) for every two open sets $W \subset C(X)$ and $U \subset X$, respectively, with $M(W) \cap U \neq \emptyset$ there exists a nonempty open $W' \subset W$ such that $M(W') \subset U$;
(d) let $\{f_0\} = f_0^{-1} B^*$, where $(B_n)_{n \geq 1}$ is a decreasing sequence of subsets in $C(X)$ with $\lim_{n \to \infty} \text{diam}(B_n) = 0$. Then $M(f_0) = \cap_{n=1}^{\infty} M(B_n)$.

Proof. (a) Let $W$ be an open subset of $C(X)$ and $x_0 \in M(f_0)$ for some $f_0 \in W$. Take $\varepsilon > 0$ such that the ball $B(f_0, \varepsilon) := \{f \in C(X): ||f - f_0|| < \varepsilon\} \subset W$. Then each $x \in B(f_0, \varepsilon)$ is a minimizer of some $f$ from $W$, e.g., of the function $f_0$ considered above.

(b) Let $x_0 \in U$. Since $X$ is completely regular, there exists $h_0 \in C(X)$ with $h_0(x_0) = 0$, $h_0(X \setminus U) = 1$, and $\|h\| \leq 1$. It is easy to see that $M(B(h_0, 1/3)) \subset U$.

(c) Let $x_0 \in M(f_0) \cap U$ for some $f_0 \in W$. Consider the function $h_0$ from (b) and find $\delta > 0$ such that $f_0 + \delta h_0 \in W$. Let further, $W' \subset W$ be an open set in $C(X)$ containing $f_0 + \delta h_0$ and such that $\text{diam}(W') < \delta/3$. Take $f \in W'$. Since for $x \in X \setminus U$ one has $f(x) \geq (f_0 + \delta h_0)(x) - \delta/3 = f_0(x) + (2\delta)/3 \geq f_0(x_0) + (2\delta)/3 = (f_0 + \delta h_0)(x_0) + (2\delta)/3 > f(x_0)$, we see that $M(f) \subset U$.

(d) Obviously $M(f_0) \subset \cap_{n=1}^{\infty} M(B_n)$. On the other hand, take some $x \in M(B_n)$. Then $x \in M(f_n)$ for some $f_n \in B_n$ with $\|f_n - f_0\| \leq \text{diam}(B_n)$. Hence, $x \in \Omega_{f_0}(2\text{diam}(B_n))$. Therefore, $M(B_n) \subset \Omega_{f_0}(2\text{diam}(B_n))$. This gives $\cap_{n=1}^{\infty} M(B_n) \subset \cap_{n=1}^{\infty} \Omega_{f_0}(2\text{diam}(B_n)) = M(f_0)$.

3. The main result

The main result in this article is the following characterization of the fact that the space $X$ possesses an $\alpha$-winning strategy.

Theorem 3.1. The space $X$ admits a winning strategy for the player $\alpha$ in the Banach-Mazur game $BM(X)$ if and only if $\text{Dom}(M)$ contains a dense and $G_\delta$-subset of $C(X)$.

Proof. Sufficiency. Suppose $\text{Dom}(M)$ contains a dense and $G_\delta$-subset of $C(X)$. Then there exist countably many open and dense subsets $(G_n)_{n \geq 1}$ of $C(X)$ such that $\cap_{n=1}^{\infty} G_n \subset \text{Dom}(M)$. The sets $F_n := C(X) \setminus G_n$, $n \geq 1$, are closed and nowhere dense in $C(X)$. That is $\text{Int}(F_n) = \emptyset$ for every $n \geq 1$.

We show that the player $\alpha$ has a winning strategy $s$ in the game $BM(X)$. Let $U_1$ be a nonempty open subset of $X$. Consider the set $\text{Int} M^*(U_1)$ that is nonempty by Proposition 2.1(b). Since $F_1$ is closed and nowhere dense in $C(X)$, the set $\text{Int} M^*(U_1) \setminus F_1$ is nonempty and open in $C(X)$. Take an open ball $B_1$ in $C(X)$ with radius less or equal to 1, such that $B_1 \subset \text{Int} M^*(U_1) \setminus F_1$. Define now the value of the strategy $s$ at $U_1$ by $s(U_1) := M(B_1)$. By Proposition 2.1(a), $s(U_1)$ is a nonempty open subset of $U_1$.

Further, let $U_2$ be an arbitrary nonempty open subset of $V_1 = s(U_1) = M(B_1)$. Since $U_2 \subset M(B_1)$ there is some $f \in B_1$ such that $M(f) \cap U_2 \neq \emptyset$. Hence, by Proposition 2.1(c) there exists a nonempty open $W \subset B_1$ such
that $M(W) \subset U_2$. As above the set $W \setminus F_2$ is a nonempty and open subset of $C(X)$. Take an open ball $B_2$ with radius less or equal to $1/2$ such that $\text{Cl}(B_2) \subset W \setminus F_2 \subset B_1$ and put $s(U_1, V_1, U_2) := M(B_2)$. Obviously $s(U_1, V_1, U_2)$ is a nonempty open subset of $U_2$. Proceeding by induction we define $s$ for every chain $(U_1, V_1, \ldots, U_n)$, $n \geq 1$, such that $U_k \subset V_{k-1}$ and $V_{k-1} = s(U_1, V_1, \ldots, U_{k-1})$ for every $k, 2 \leq k \leq n$.

Let $U_1 \supset V_1 \supset \cdots \supset U_n \supset V_n \supset \cdots$ be an infinite sequence of open sets in $X$ such that for every $n \geq 1$, $V_n = s(U_1, V_1, \ldots, U_n)$. Let $(B_n)_{n \geq 1}$ be the sequence of open balls in $C(X)$ associated with $(U_n)_{n \geq 1}$ and $(V_n)_{n \geq 1}$ from the construction of $s$. Then for every $n \geq 1$:

1. $\text{Cl}(B_{n+1}) \subset B_n$ and $B_n \cap F_n = \emptyset$;
2. $\text{diam}(B_n) < 1/n$;
3. $V_n = M(B_n)$.

Conditions (1) and (2) guarantee that $\bigcap_{n=1}^{\infty} B_n$ is a one-point set in $C(X)$, say $f_0$. Moreover, (1) shows in addition that $f_0 \in C(X) \setminus \bigcup_{n=1}^{\infty} F_n \subset \text{Dom}(M)$. Therefore, by (3) and Proposition 2.1(d) we have

$$\emptyset \neq M(f_0) = \bigcap_{n=1}^{\infty} M(B_n) = \bigcap_{n=1}^{\infty} V_n.$$ 

Hence $s$ is a winning strategy for the player $\alpha$ in the Banach-Mazur game $BM(X)$.

Necessity. Suppose now, that there exists a winning strategy $s$ for the player $\alpha$ in the game $BM(X)$. Every finite sequence of sets $(U_1, V_1, \ldots, U_n, V_n)$, $n \geq 1$, obtained by the first $n$ steps in the game $BM(X)$ is called a partial play in this game. The key step in the proof is the following lemma:

**Lemma 3.2.** Let $(U_1, V_1, \ldots, U_n, V_n)$, $n \geq 1$, be a partial play in the game $BM(X)$ and $W_n$ be a nonempty open subset of $C(X)$ such that $M(W_n) \subset V_n$. Then there is a family $\Gamma(W_n)$ of triples $(U_{n+1}, V_{n+1}, W_{n+1})$ such that:

(a) $U_{n+1}$ is a nonempty open subset of $M(W_n)$;
(b) $V_{n+1} = s(U_1, V_1, \ldots, U_n, V_n, U_{n+1})$;
(c) $W_{n+1}$ is a nonempty subset of $C(X)$ such that $\text{diam}(W_{n+1}) < 1/(n + 1)$,
Cl$(W_{n+1}) \subset W_n$, and $M(W_{n+1}) \subset V_{n+1}$;
(d) the family $\gamma(W_n) := \{W_{n+1} : (U_{n+1}, V_{n+1}, W_{n+1}) \in \Gamma(W_n) \text{ for some } U_{n+1}, V_{n+1}\}$ is disjoint;
(e) the set $H(W_n) := \bigcup\{W_{n+1} : W_{n+1} \in \gamma(W_n)\}$ is dense in $W_n$.

**Proof of Lemma 3.2.** Take a maximal family $\Gamma(W_n)$ satisfying the properties (a)–(d). We prove that it satisfies also the condition (e).

Suppose the contrary. There exists a nonempty open subset $G$ of $C(X)$ with $G \subset W_n$ and $G \cap H(W_n) = \emptyset$. Consider the set $M(G)$ that, by Proposition 2.1(a), is a nonempty and open subset of $X$. Moreover, $M(G) \subset M(W_n) \subset V_n$. Let $U_{n+1} := M(G)$ and $V_{n+1} := s(U_1, V_1, \ldots, U_n, V_n, U_{n+1})$. By Proposition 2.1(c) there is a nonempty open subset $W_{n+1}$ of $C(X)$ such that $W_{n+1} \subset G$ and $M(W_{n+1}) \subset V_{n+1}$. We may arrange, in addition, $\text{Cl}(W_{n+1}) \subset W_n$ and $\text{diam}(W_{n+1}) < 1/(n + 1)$. Now, the family $\Gamma' := \Gamma(W_n) \cup \{(U_{n+1}, V_{n+1}, W_{n+1})\}$ is strictly larger than $\Gamma(W_n)$ and satisfies (a)–(d). This is a contradiction showing that the maximal family $\Gamma(W_n)$ satisfies also (e). □
Let us mention that Lemma 3.2 is true also for $n = 0$ provided we put $U_0 = V_0 = X$. Now, let us get back to the proof of the theorem. We proceed in the following way.

Put $\gamma_0 := \{C(X)\}$, $W_0 = C(X)$, $U_0 = V_0 = X$ and apply Lemma 3.2 for the triple $(U_0, V_0, W_0)$. We get a family of triples $\Gamma_1 := \Gamma_1(W_0)$ satisfying conditions (a)–(e) from Lemma 3.2. Put $\gamma_1 := \gamma(W_0)$ and $H_1 := H(W_0)$. By (e) the set $H_1$ is open and dense in $C(X)$. Further, because of (d), for every $W_1 \in \gamma_1$ there is a unique couple $(U_1, V_1)$ with $(U_1, V_1, W_1) \in \Gamma_1$. Apply again Lemma 3.2 for this triple. As a result, for every $W_1 \in \gamma_1$ we obtain a family of triples $\Gamma(W_1)$ with the properties (a)–(e) fulfilled with respect to the couple $(U_1, V_1)$ corresponding to $W_1$. Let $\Gamma_2 := \bigcup \{\Gamma(W_1): W_1 \in \gamma_1\}$, $\gamma_2 := \bigcup \{\gamma(W_1): W_1 \in \gamma_1\}$, and $H_2 := \bigcup \{H(W_1): W_1 \in \gamma_1\}$. Since $\gamma_1$ is disjoint and each $\gamma(W_1)$ is disjoint too, then the family $\gamma_2$ is also disjoint. Moreover, by (e) every $H(W_1)$ is dense in $W_1$ and since $H_1$ is dense in $C(X)$, it follows that $H_2$ is (open) and dense in $C(X)$ as well.

Proceeding in this way we obtain a sequence of families $(\Gamma_n)_{n \geq 1}$ of triples and a sequence of disjoint families $(\gamma_n)_{n \geq 0}$ of open sets in $C(X)$, with $\gamma_0 = \{C(X)\}$, such that for every $n \geq 1$ we have

(i) $\Gamma_n$ is a union of the families $\Gamma(W_{n-1})$, $W_{n-1} \in \gamma_{n-1}$, where $\Gamma(W_{n-1})$ is obtained by Lemma 3.2 from some uniquely determined partial play $(U_1, V_1, \ldots, U_{n-1}, V_{n-1})$;

(ii) $\gamma_n$ is a union of the families $\gamma(W_{n-1})$, $W_{n-1} \in \gamma_{n-1}$ from the condition (d) of Lemma 3.2;

(iii) the set $H_n := \bigcup \{W_n: W_n \in \gamma_n\}$ is open and dense in $C(X)$.

Let $H_0 := \bigcap_{n=1}^{\infty} H_n$. Obviously $H_0$ is a dense and $G_\delta$-subset of $C(X)$. Take $f_0 \in H_0$. By the properties above, this $f_0$ determines a unique sequence $(W_n)_{n \geq 1}$ such that for every $n \geq 1$, $W_n \in \gamma_n$, $\text{Cl}(W_{n+1}) \subset W_n$, and $\text{diam}(W_n) < 1/n$. Hence $\{f_0\} = \bigcap_{n=1}^{\infty} W_n$. By the properties (a)–(d) from Lemma 3.2 and conditions (i)–(iii) above it follows that there is an infinite sequence of open sets

$$U_1 \supset V_1 \supset \cdots \supset U_n \supset V_n \supset \cdots$$

such that $V_n = s(U_1, V_1, \ldots, U_n)$ and $U_{n+1} \subset M(W_n) \subset V_n$ for every $n \geq 1$.

Hence, by Proposition 2.1(d) we have

$$M(f_0) = \bigcap_{n=1}^{\infty} M(W_n) = \bigcap_{n=1}^{\infty} V_n = \bigcap_{n=1}^{\infty} U_n.$$ 

Since $s$ is a winning strategy, we see that $M(f_0) = \bigcap_{n=1}^{\infty} V_n \neq \emptyset$. The proof is complete. $\square$

As an immediate corollary from Theorem 3.1 we get the following sufficient condition for the set $\text{Dom}(M)$ to be residual in $C(X)$.

**Corollary 3.3** (see [St, Theorem 5]). Let $X$ admit an $\alpha$-winning tactic in the Banach-Mazur game. Then the set $\text{Dom}(M)$ contains a dense and $G_\delta$-subset of $C(X)$.

As mentioned above, there exists a completely regular space $X$ with $\alpha$-winning strategy that does not admit an $\alpha$-winning tactic (see [De]).
A slight change in the proof of Theorem 3.1 gives us the possibility to characterize the spaces $X$ for which the set $\{f \in C(X) : f$ attains its minimum in $X$ at exactly one point$\}$ contains a dense and $G_\delta$-subset of $C(X)$, i.e., to characterize the spaces $X$ in which we have generic uniqueness of the solution of the minimization problems generated by functions from $C(X)$.

**Theorem 3.4.** The set $\{f \in C(X) : f$ attains its minimum in $X$ at exactly one point$\}$ contains a dense and $G_\delta$-subset of $C(X)$ if and only if the space $X$ admits an $\alpha$-winning strategy $s$ such that, whenever one has a sequence of open sets $U_1 \supset V_1 \supset \cdots \supset U_n \supset V_n \supset \cdots$ with $V_n = s(U_1, V_1, \ldots, U_n)$ for every $n \geq 1$, then $\bigcap_{n=1}^{\infty} V_n$ is a one-point set.

Let us mention a class of spaces for which we have generic uniqueness of the solution for the minimization problems for the functions from $C(X)$. The topological space $X$ is called fragmentable (see [JaRo]) if there is a metric $\rho$ on it such that for every $\epsilon > 0$ and every nonempty subset $Y$ of $X$ there exists a nonempty relatively open subset $A$ of $Y$ with $\rho$-diam$(A) < \epsilon$. Further information about fragmentable spaces can be found in [Na, Ri].

**Corollary 3.5.** Let $X$ be a fragmentable space that admits an $\alpha$-winning strategy. Then the set $\{f \in C(X) : f$ attains its minimum in $X$ at exactly one point$\}$ contains a dense and $G_\delta$-subset of $C(X)$.

**Proof.** Let $\rho$ be the metric on $X$ that fragments it. Let $U_1, V_1, \ldots, U_n$ be the first $n$ steps of the player $\beta$ and the first $n - 1$ steps of $\alpha$ in the game $BM(X)$, $n \geq 1$. Take a nonempty open set $U'_n \subset U_n$ such that $\rho$-diam$(U'_n) < 1/n$, and define $s'(U_1, V_1, \ldots, U'_n) = s(U_1, V_1, \ldots, U'_n)$. It is easy to check that the so obtained strategy $s'$ satisfies the requirements of Theorem 3.4. □

We give some further corollaries of Theorem 3.1. Before that let us recall some notions. A minimization problem generated by some $f \in C(X)$ (which we will denote by $(X, f)$) is said to be Tikhonov well posed (see [Ti]) if it has unique solution $x_0 \in X$ and, moreover, every minimizing sequence $(x_n)^{\infty}_{n=1}$ (i.e., $f(x_n) \to \inf(X, f)$) converges to this unique solution. If $(X, f)$ is Tikhonov well posed then every minimizing net (not only every minimizing sequence) converges to its unique solution (see, e.g., [ČKR]). Let

$$T := \{f \in C(X) : (X, f)$ is Tikhonov well posed$\}.$$

The following fact is proved for compact spaces $X$ in [ČK1, ČK2] and for an arbitrary $X$ in [ČKR, Theorem 3.5]: The set $T$ contains a dense and $G_\delta$-subset of $C(X)$ iff the space $X$ contains a dense and completely metrizable subspace.

A family $\gamma$ of subsets of $X$ is called a net in $X$ if for every $x \in X$ and every open $U \subset X$, with $x \in U$, there exists $H \in \gamma$ such that $x \in H \subset U$. The space $X$ possesses a $\sigma$-discrete net if there are countably many discrete families $(\gamma_n)^{\infty}_{n=1}$ in $X$ such that $\gamma := \bigcup_{n=1}^{\infty} \gamma_n$ forms a net in $X$. Recall that a family of subsets in $X$ is discrete if every point in $X$ has a neighborhood that intersects at most one element of the family. Every metric space has a $\sigma$-discrete net. The following is again a result proved in [ČKR, Theorem 5.6]: Suppose that $X$ possesses a $\sigma$-discrete net. Then the set $T \cup (C(X) \setminus \text{Dom}(M))$ contains a dense and $G_\delta$-subset of $C(X)$. 


In the special case when $X$ has a $\sigma$-discrete net we can give another characterization of the fact that the space $X$ admits an $\alpha$-winning strategy. For a metric space $X$ this characterization follows by a result of [Ox] and for a space $X$ with a base of countable order can be found in [Wh].

**Theorem 3.6.** Let $X$ possess a $\sigma$-discrete net. Then $X$ admits an $\alpha$-winning strategy in the game $BM(X)$ if and only if $X$ contains a dense and completely metrizable subspace.

**Proof.** The proof is an immediate consequence from Theorem 3.1 and the mentioned results from [ČKR]. $\square$

Every space $X$ that contains a dense and completely metrizable subspace possesses $\alpha$-winning tactic (see, e.g., [Tel]). Then we have

**Corollary 3.7.** Let $X$ possess a $\sigma$-discrete net. Then $X$ admits $\alpha$-winning strategy in the game $BM(X)$ if and only if it admits $\alpha$-winning tactic in this game.

**References**


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