SHORE POINTS AND DENDRITES

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Abstract. A point \( x \) in a dendroid \( X \) is called a shore point if there is a sequence of subdendroids of \( X \) not containing \( x \) and converging to \( X \) in the Hausdorff metric. We give necessary and sufficient conditions for a dendroid to be a dendrite, in terms of shore points and Kelley's property.

INTRODUCTION

A dendroid is an arcwise connected, hereditarily unicoherent metric continuum. A locally connected dendroid is called a dendrite. It is well known that every pair of points \( u \) and \( w \) in a dendroid are joined by a unique arc \([u, w]\) and that the subcontinua of a dendroid are themselves dendroids. If \( X \) is a dendroid and \( x \in X \), then \( x \) is an end point of \( X \) if it is an end point of every arc containing it, and \( x \) is a shore point of \( X \) if there exists a sequence \( \{X_n\} \) of subdendroids of \( X \) not containing \( x \) such that \( \lim X_n = X \).

It is not difficult to prove that every end point is a shore point. The shore points of \( X \) that are not end points will be called the improper shore points of \( X \). The following example shows that a dendroid without improper shore points is not necessarily a dendrite: Let \( X \subseteq \mathbb{R}^2 \) be the union of the rectilinear segments \( [(0,0), (1,1/n)] \), \( n = 1, 2, 3, \ldots \) and \( [(0,0), (2,0)] \).

A dendroid will be called neat whenever each one of its subdendroids has no improper shore points. Obviously every subdendroid of a neat dendroid is neat.

In Theorem 2.1 we give necessary and sufficient conditions for a dendroid \( X \) to be a dendrite in terms of shore points and Kelley's property. In particular, it is proved that \( X \) is neat iff \( X \) is a dendrite.

1. Preliminaries

A dendroid \( X \) is smooth at \( p \) if \([p, a_n]\) converges to \([p, a]\) in the Hausdorff metric, provided \( a_n \) converges to \( a \) in \( X \) (see [2]). A continuum \( X \) has Kelley's property if for every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that for every pair of points \( a \) and \( b \) in \( X \) whose distance is less than \( \delta \) and each subcontinuum
A of X containing a, there is a subcontinuum B of X containing b whose Hausdorff distance from A is less than \( \varepsilon \) [4]. Recently, Czuba [1] has proved the following result:

1.1. **Theorem** (Czuba). If a dendroid has Kelley's property then it is smooth.

For general terminology we refer the reader to [4, 6]. A weaker version of the following lemma was proved in [5]. The proof is not difficult and is actually identical to the previous one.

1.2. **Lemma.** If \( U \) is an arcwise connected subset of a dendroid X then \( \text{Cl}(U) \) is the limit of a sequence of subdendroids of X contained in \( U \).

1.3. **Lemma.** Let X be a dendroid that has Kelley's property. Then for every \( p \in X \) and every arc-component \( U \) of \( X\{p\} \), either \( U \) is open or \( \text{Int}(U) = \emptyset \).

**Proof.** Suppose that \( \text{Int}(U) \neq \emptyset \) and let \( v \in \text{Int}(U) \). If \( u \in U \setminus \text{Int}(U) \) then the arc \( [u, v] \subseteq U \). Let \( 0 < \varepsilon < \min\{d(p, [v, u]), \alpha\} \) where d denotes the distance in X and the ball of radius \( \alpha \) centered at \( v \) is contained in \( \text{Int}(U) \). For each \( \delta > 0 \), there exists \( w \notin U \) such that \( d(w, u) < \delta \).

Let \( K \) be a subcontinuum of X containing \( w \): If \( p \notin K \) then \( K \) is contained in an arc-component of \( X\{p\} \) different from \( U \), so that \( d(v, K) \geq \alpha > \varepsilon \), which implies \( D(K, [u, v]) > \varepsilon \), where \( D \) denotes the distance in the Hausdorff metric. If \( p \in K \) then \( d(p, [v, u]) > \varepsilon \) and again \( D(K, [v, u]) > \varepsilon \). Therefore, Kelley's property is not satisfied. \( \square \)

1.4. **Lemma.** A shore point in a dendroid X is not a cut point of X.

**Proof.** Suppose that for some \( q \in X \), \( X\{q\} = H \cup K \) is a decomposition of \( X\{q\} \) into disjoint, relatively closed sets \( H \) and \( K \) and, let \( \varepsilon = D(H, K) \). If for a subcontinuum \( A \) of X, \( D(A, X) < \varepsilon \), then the sets \( A \cap H \) and \( A \cap K \) are nonempty, so that \( q \in A \). Therefore, \( q \) is not a shore point of X. \( \square \)

2. **Main result**

2.1. **Theorem.** For a dendroid X, the following conditions are equivalent:

(i) \( X \) is neat.

(ii) For every \( q \in X \), the arc components of \( X\{q\} \) are all open.

(iii) \( X \) is a dendrite.

(iv) \( X \) has Kelley's property and has no improper shore points.

(v) Every subcontinuum of X has Kelley's property.

**Proof.** (i) \( \Rightarrow \) (ii). Suppose that an arc component \( \alpha \) of \( X\{q\} \) is not open. If for some arc component \( \beta \) of \( X\{q\} \) different from \( \alpha \), \( \text{Cl}(\beta) \cap \alpha \neq \emptyset \), we take \( x \in \text{Cl}(\beta) \cap \alpha \) and note that the arc \( (q, x] \subseteq \text{Cl}(\beta) \cap \alpha \). If \( y \in (q, x) \), then \( y \in \text{Cl}(\beta \setminus \alpha) \), so that there exists a sequence \( \{X_n\} \) of subdendroids contained in \( \beta \) such that \( X_n \to \text{Cl}(\beta) \) (Lemma 1.2). Clearly \( y \) is an improper shore point of the subdendroid \( \text{Cl}(\beta) \). Let \( \Gamma \) be the set of arc components of \( X\{q\} \) different from \( \alpha \). We suppose now that \( \text{Cl}(\beta) \cap \alpha = \emptyset \) for every \( \beta \in \Gamma \) and denote by \( B \) the union of the members of \( \Gamma \). By assumption \( \text{Cl}(B) \cap \alpha \neq \emptyset \), take \( x \in \text{Cl}(B) \cap \alpha \) and \( y \in (q, x) \). Notice that \( y \in \text{Cl}(B \setminus \alpha) \). Let \( \{y_n\} \) be a sequence of points such that \( y_n \in B \setminus \alpha \) and \( \{y_n\} \) converges to \( y \) in \( \text{Cl}(B) \setminus \alpha \). We can assume that \( \beta_n \neq \beta_m \) for \( m \neq n \). The sequence of dendroids \( M_n = \bigcup_{j=1}^{n} \text{Cl}(\beta_j) \) is increasing and satisfies \( M_n \cap \alpha = \emptyset \).
for each \( n \). Moreover, \( \{M_n\} \) converges to a subdendroid \( Y \subseteq \text{Cl}(B) \) and hence \( y \) is an improper shore point of \( Y \).

(ii) \( \Rightarrow \) (i). Suppose that \( X \) is not neat. Let \( X_0 \) be a subdendroid of \( X \) and \( q \) an improper shore point of \( X_0 \). Then \( X_0 \setminus \{q\} \) has at least two arc components.

We shall prove that every arc component \( \alpha \) of \( X_0 \setminus \{q\} \) is open in \( X_0 \setminus \{q\} \). Since this fact contradicts the connectivity of \( X_0 \setminus \{q\} \), our assertion follows from 1.4. Indeed if \( C(\alpha) \) is the arc component in \( X \setminus \{q\} \) containing \( \alpha \), then \( C(\alpha) \cap (X_0 \setminus \{q\}) = \alpha \).

(i) \( \Rightarrow \) (iii). It was proved by Charatonik and Eberhart [2, Corollaries 4 and 5] that a dendroid \( X \) is a dendrite iff \( X \) is smooth at each of its points. Suppose that \( X \) is not smooth at \( q \), and let \( \{x_n\} \) be a sequence that converges to \( x \) such that \([q, x_n]\) is convergent but \( L = \lim_{n \to \infty} [q, x_n] \neq [q, x] \). Let \( z \in L \setminus [q, x] \) be a point that is not an end point of \( L \). If \( z \notin [q, x_n] \) for an infinite set \( J \) of indices, it will be clear that \( z \) is an improper shore point of \( X \setminus \{q, x, \alpha(\{q, x\}) \} \).

Therefore we can assume that \( z \in [q, x_n] \) for all \( n \). In \( X \setminus \{z\} \), the arcs \([q, z] \) and \([x_n, z] \) belong to different arc components \( \alpha([q, z]) \) and \( \alpha([x_n, z]) \), respectively. Since (i) implies (ii), it follows that \( \alpha([q, z]) \) and \( \bigcup_n \alpha([x_n, z]) \) are open. Moreover, they are disjoint, which is impossible since \( x \in \alpha([q, z]) \) and \( x_n \to x \).

(i) \( \Rightarrow \) (iv). This follows from (i) \( \Rightarrow \) (iii) since every locally connected continuum has Kelley's property.

(iv) \( \Rightarrow \) (ii). Suppose that for some \( p \in X \), \( X \setminus \{p\} \) has a nonopen arc component \( U \). Let \( u \) be a non end point of \( X \) contained in \( U \) and for each \( n \in \mathbb{N} \), let \( C_n \) be the component of \( X \setminus B_{1/n}(u) \) containing \( p \). If \( x \in X \setminus U \) then \( [p, x] \cap U = \emptyset \). In particular, \( u \notin [p, x] \), so that for \( n \) large enough \([p, x] \cap B_{1/n}(u) = \emptyset \). This implies that \( [p, x] \subseteq C_n \). By Lemma 1.3, \( \text{Int}(U) = \emptyset \), so that \( C_n = X \). Since \( u \notin C_n \) for every \( n \), it follows that \( u \) is an improper shore point of \( X \).

(v) \( \Rightarrow \) (ii) By Theorem 1.1 \( X \) is smooth. By [3, Theorem 1, p. 194] \( X \) contains no subdendroid of Type 1. Next we show that \( X \) is smooth at each of its points. Let \( p \in X \) and suppose \( X \) is not smooth at \( p \). By [3, Lemma 1, p. 193], \( X \) contains a subdendroid of Type 3. But a Type 3 dendroid contains a subdendroid that does not have Kelley's property, a contradiction. By [2] \( X \) is a dendrite and a dendrite clearly satisfies (ii).

(iii) \( \Rightarrow \) (v). This follows since every subdendroid of a dendrite is a dendrite.

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References


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