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C^*-EXTREME POINTS OF SOME COMPACT C^*-CONVEX SETS

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ABSTRACT. In the C^*-algebra M_n of complex $n \times n$ matrices, we consider the notion of noncommutative convexity called C^*-convexity and the corresponding notion of a C^*-extreme point. We prove that each irreducible element of M_n is a C^*-extreme point of the C^*-convex set it generates, and we classify the C^*-extreme points of any C^*-convex set generated by a compact set of normal matrices.

1. INTRODUCTION

For C^*-algebras and, more generally, for bimodules over C^*-algebras there is a notion of convexity that incorporates algebra-valued convex coefficients in a natural way. This form of convexity, called C^*-convexity, was studied in its own right by Loebl and Paulsen in [6] and by Hopenwasser, Moore, and Paulsen in [5]. There is an extremal theory associated with C^*-convexity and it is our aim in this paper to further its development by studying the C^*-extreme points of specific compact C^*-convex sets in the C^*-algebra M_n of complex $n \times n$ matrices.

Sections 3 and 4 contain the main results. In §3 it is proved that every irreducible element of M_n is a C^*-extreme point of the C^*-convex set it generates. In §4 conditions are presented that must be satisfied by the C^*-extreme points of a C^*-convex set generated by a family of matrices; these necessary conditions lead to a classification of the C^*-extreme points of compact C^*-convex sets generated by normals. Auxiliary results include an extension of Stampfli's theorem concerning the extreme points of the classical numerical range of a hyponormal operator and a result to the effect that irreducible elements $x \in M_n$ can be determined up to unitary equivalence among all elements of M_n by the norms of linear polynomials in x with coefficients taken from M_n.

We begin by recalling the definitions [6] of C^*-convexity and C^*-extreme point. A subset K of a unital C^*-algebra A is said to be C^*-convex if K is closed under the formation of finite sums of the type $\sum_i t_i^* x_i t_i$, where each

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$x_i \in K$ and the elements $t_i \in A$ are under the proviso that $\sum_i t_i^* t_i = 1$. The t_i are called C^*-convex coefficients and the C^*-convex combination $\sum_i t_i^* x_i t_i$ is called proper if each coefficient t_i is invertible in A. An element x in a C^*-convex set $K \subset A$ is said to be a C^*-extreme point of K if the only possible representations of x as proper C^*-convex combinations of elements $x_i \in K$ are those for which each x_i comes from the unitary orbit of x. To distinguish extreme points from C^*-extreme points, the former are sometimes referred to as linear extreme points.

In [6] it is shown that the C^*-extreme points of C^*-convex subsets of M_n are linear extreme points, and in [5] an example in M_2 of a C^*-convex set having some linear extreme points that are not C^*-extreme is provided. In §4 we present a new class of examples, occurring in every matrix algebra, that show that linear extreme points need not be C^*-extreme. Although a compact C^*-convex subset of M_n always possesses C^*-extreme points [4], the Krein-Milman type problem of whether such extreme points exist in sufficient numbers to recover the original set remains open.

Complete information about the C^*-extreme points of certain C^*-convex sets is available in a few special cases: the closed unit ball of the algebra of operators acting on a Hilbert space, the closed interval $\{x : 0 \leq x \leq 1\}$ of operators, and the set of $x \in M_2$ with numerical radius $w(x) \leq 1$ (see [5]). We augment these results by completing the classification begun in [4] of the C^*-extreme points of C^*-convex sets generated by compact sets of normal matrices (Corollary 4.2) and by determining all of the C^*-extreme points of the C^*-convex set generated by a single 2×2 matrix (Theorem 4.3).

One source for interest in C^*-convexity is the theory of matricial ranges [1, §2.4; 3; 9, §4]. The nth matricial range of an element a in a unital C^*-algebra A is the compact C^*-convex subset $W^n(a) \subset M_n$ of elements of the form $\phi(a)$, where ϕ denotes a unital completely positive map of A into M_n. If x is an element of $B(H)$, the C^*-algebra of bounded linear operators on a complex Hilbert space H, then the numerical range $W^1(x)$ of x is the closure of the classical numerical range $\{(x\xi, \xi) : \xi \in H, \|\xi\| = 1\}$. In §5 some of our results on C^*-extreme points are applied to the matricial ranges of hyponormal operators.

The C^*-convex hull of a subset $\mathscr{S} \subset M_n$ is the smallest C^*-convex set containing $\mathscr{S}$ and is denoted by C^*-conv $\mathscr{S}$. An essential fact is that C^*-conv $\mathscr{S}$ is compact whenever $\mathscr{S}$ is a compact subset of M_n [3, 2.4].

The unitary equivalence of x and z is denoted by $x \sim z$ and is taken to mean that there is a unitary u such that $x = u^* z u$.

2. On a complete unitary invariant of irreducible matrices

The (finite) order of an irreducible operator $x \in B(H)$ is the least positive integer k, if it exists, for which the values of $\| \sum_{i=0}^k x^i \otimes c_i \|$, where $c_0, c_1, \ldots, c_k$ are complex $n \times n$ matrices with n arbitrary, determine x among the irreducible elements of $B(H)$ uniquely up to unitary equivalence. An important theorem of Arveson asserts that for irreducible compact operators acting on a complex separable Hilbert space, the order exists and is equal to 1 [1, Chapter 2]; in particular, irreducible elements in the C^*-algebra M_n are of first order. The purpose of this section is to show that for the algebra M_n the situation is

somewhat more special in that irreducible elements $x \in M_n$ can be determined up to unitary equivalence among *all* elements of M_n by the norms of linear matrix polynomials $x \otimes c_1 + 1 \otimes c_0$ in just one algebra only—$M_n \otimes M_n$.

2.1. **Theorem.** *Suppose that* $x, z \in M_n$ *satisfy* $\|x \otimes b + 1 \otimes m\| = \|z \otimes b + 1 \otimes m\|$ *for every* $b, m \in M_n$. *If* x *is irreducible, that is, if the only selfadjoint idempotents commuting with* x *are* 0 *and* 1, *then* z *must be irreducible and, moreover, unitarily equivalent to* x.

As in Arveson's paper [1], the proof of Theorem 2.1 involves the use of his boundary theorem and the comparison of the matricial ranges of z with those of x. It is in this latter aspect that the connection with C^*-convexity theory arises.

Every unital completely positive map $\phi : M_n \to M_n$ has the form $\phi(\cdot) = \sum_i t_i^*(\cdot)t_i$, where the sum is a finite sum and the matrices t_i satisfy $\sum_i t_i^* t_i = 1$ [2]. Hence, the C^*-convex set generated by $x \in M_n$ is precisely the nth matricial range of x, and so we are able to make use of some of the results of [3; 9] in a manner summarized by the

2.2. **Theorem.** *For* $x, z \in M_n$, *the following statements are equivalent.*

(1) $C^*\text{-conv}\{x\} \subset C^*\text{-conv}\{z\}$;
(2) $W^1(x \otimes b + 1 \otimes m) \subset W^1(z \otimes b + 1 \otimes m)$ *for every* $b, m \in M_n$;
(3) $\|x \otimes b + 1 \otimes m\| \le \|z \otimes b + 1 \otimes m\|$ *for every* $b, m \in M_n$.

Proof. In identifying C^*-conv $\{x\}$ with $W^n(x)$ and C^*-conv $\{z\}$ with $W^n(z)$, the implication (3) $\Rightarrow$ (1) is given by [3, 2.1] and the equivalence of (1) and (2) by [9, 4.3]. To prove the implication (1) $\Rightarrow$ (3), observe that because $x \in W^n(x) \subset W^n(z)$, there is a unital completely positive $\phi : M_n \to M_n$ with $x = \phi(z)$. The induced map $\phi \otimes \mathrm{id}$ on $M_n \otimes M_n$, where $\mathrm{id} : M_n \to M_n$ is the identity map, is a contraction, and therefore for each $b, m \in M_n$, $\|x \otimes b + 1 \otimes m\| = \|\phi(z) \otimes b + 1 \otimes m\| \le \|z \otimes b + 1 \otimes m\|$. □

We now provide the

Proof of Theorem 2.1. Using Theorem 2.2, the hypothesis is that $W^n(x) = W^n(z)$. There exist, therefore, unital completely positive maps $\phi, \psi : M_n \to M_n$ such that $x = \psi(z)$ and $z = \phi(x)$, whence $x = \psi \circ \phi(x)$. Let $\mathscr{L}$ denote the subspace spanned by x, x^*, and 1, and let ω denote the identity map $\mathscr{L} \to M_n$: namely, $\omega = \psi \circ \phi|_{\mathscr{L}}$. Because $\mathscr{L}$ is irreducible and the C^*-algebra $\mathscr{L}$ generates is M_n, the boundary theorem of Arveson [1, 2.1.1] states that ω has a unique completely positive extension to M_n. The identity map and $\psi \circ \phi$, being two such extensions of ω, must therefore coincide.

From $\mathrm{id} = \psi \circ \phi$, it follows that, for every $a \in M_n$,

$$\|a\| = \|\psi(\phi(a))\| \le \|\phi(a)\| \le \|a\| \ ;$$

that is, ϕ is a unital isometry on M_n and, therefore, has the form $x \mapsto u^*xu$ or $x \mapsto u^*x^tu$ for some unitary $u \in M_n$ [7]. Complete positivity rules out the latter possibility, so that $z = \phi(x) = u^*xu$ as desired. □

3. C^*-CONVEX SETS WITH A SINGLE, IRREDUCIBLE GENERATOR

Using the main result of §2, we will prove that each irreducible element of M_n is C^*-extreme in the C^*-convex set it generates.

3.1. **Theorem.** *If $x \in M_n$ is irreducible, then x is a C^*-extreme point of its C^*-convex hull.*

Proof. Suppose that $x = \sum_i t_i^* x_i t_i$ for some elements $x_i \in C^*\text{-conv}\{x\}$ and for some invertible C^*-convex coefficients $t_i \in M_n$. Because each $x_i \in C^*$-conv$\{x\}$, it follows from (2) of Theorem 2.2 that $W^1(x_i \otimes b + 1 \otimes m) \subset W^1(x \otimes b + 1 \otimes m)$ for every i and for every $b, m \in M_n$. We now show that the reverse inclusions $W^1(x \otimes b + 1 \otimes m) \subset W^1(x_i \otimes b + 1 \otimes m)$ hold.

Fix $b, m \in M_n$. Let λ be an arbitrary extreme point of $W^1(x \otimes b + 1 \otimes m)$; so λ is determined by some unit vector $\xi \in \mathbb{C}^n \otimes \mathbb{C}^n$. By using the unit vectors

$$\eta_i = \|(t_i \otimes 1)\xi\|^{-1}(t_i \otimes 1)\xi$$

and the induced elements

$$\lambda_i = \big((x_i \otimes b + 1 \otimes m)\eta_i, \eta_i\big) ,$$

the extreme point λ can be expressed as a (proper) convex combination of the $\lambda_i \in W^1(x_i \otimes b + 1 \otimes m)$ as follows:

$$\begin{aligned}\lambda = \big((x \otimes b + 1 \otimes m)\xi, \xi\big) &= \sum_i \big((x_i \otimes b + 1 \otimes m)(t_i \otimes 1)\xi, (t_i \otimes 1)\xi\big) \\ &= \sum_i \|(t_i \otimes 1)\xi\|^2 \lambda_i .\end{aligned}$$

But each λ_i is an element of $W^1(x \otimes b + 1 \otimes m)$ and therefore, because λ is an extreme point of $W^1(x \otimes b + 1 \otimes m)$, we must have $\lambda_i = \lambda$ for every i. This proves that the extreme points of $W^1(x \otimes b + 1 \otimes m)$ lie in each of the sets $W^1(x_i \otimes b + 1 \otimes m)$. Hence, $W^1(x \otimes b + 1 \otimes m)$ is contained in each $W^1(x_i \otimes b + 1 \otimes m)$ and so, by Theorem 2.2, $C^*\text{-conv}\{x_i\} = C^*\text{-conv}\{x\}$ for every i. Equivalently, $\|x_i \otimes b + 1 \otimes m\| = \|x \otimes b + 1 \otimes m\|$ for every $b, m \in M_n$ and every i. But x is irreducible, and so as Theorem 2.1 asserts, each x_i is unitarily equivalent to x; that is, x is a C^*-extreme point of its C^*-convex hull. □

4. The search for C^*-extreme points; narrowing down the possibilities

Before we can hope to answer the Krein-Milman question of whether the C^*-extreme points are sufficient to recover the original C^*-convex set, it would help to be able to describe all the C^*-extreme points. The purpose of this section is to present necessary conditions for points to be C^*-extreme in various cases in order to narrow the search considerably. The main results are Theorem 4.1 and its applications.

We begin by presenting techniques for rewriting C^*-convex combinations.

Technique A. Let $x = \sum_{i=1}^m t_i^* x_i t_i$ with $\sum_i t_i^* t_i = 1$ and $\|t_i\| < 1$ for each i. If this is not already a proper C^*-convex combination, then we can rewrite it as one in the following manner. Let

$$a_i = (m-1)^{-1/2}(1 - t_i^* t_i)^{1/2}$$

(invertible since $\|t_i\| < 1$),

$$\kappa_{ij} = \frac{1 - \delta_{ij}}{m-1}, \qquad h_{ij} = \sqrt{k_{ij}}\, t_j a_i^{-1}.$$

For each i,

$$\begin{aligned}\sum_j h_{ij}^* h_{ij} &= \sum_j \sqrt{\kappa_{ij}}(a_i^{-1})^* t_j^* t a_i^{-1}\sqrt{\kappa_{ij}} \\ &= (1-t_i^*t_i)^{-1/2}\Big(\sum_j (1-\delta_{ij})t_j^*t_j\Big)(1-t_i^*t_i)^{-1/2} \\ &= (1-t_i^*t_i)^{-1/2}(1-t_i^*t_i)(1-t_i^*t_i)^{-1/2} = 1,\end{aligned}$$

and so $z_i = \sum_{j=1}^m h_{ij}^* x_j h_{ij} \in C^*\text{-conv}\{x_1, \dots, x_m\}$ by the above computation. Further, $x = \sum_i a_i^* z_i a_i$ is a representation of x as a proper C^*-convex combination of the z_i. An important subtlety that will be essential later is that each $h_{ii} = 0$ and so each z_i is a C^*-convex combination of the x_j using fewer than m nonzero coefficients.

Technique B. If $p_1^*p_1 + p_2^*p_2 = 1$ and if

$$x = \begin{pmatrix} 1 & 0 \\ 0 & p_1^* \end{pmatrix} x_1 \begin{pmatrix} 1 & 0 \\ 0 & p_1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & p_2^* \end{pmatrix} x_2 \begin{pmatrix} 0 & 0 \\ 0 & p_2 \end{pmatrix},$$

then for every scalar $\lambda \in (0, 1)$, x can be rewritten as

$$\begin{aligned} x = &\begin{pmatrix} \lambda 1 & 0 \\ 0 & \lambda p_1^* \end{pmatrix} x_1 \begin{pmatrix} \lambda 1 & 0 \\ 0 & \lambda p_1 \end{pmatrix} \\ &+ \begin{pmatrix} \sqrt{1-\lambda^2}1 & 0 \\ 0 & (1-\lambda^2 p_1^*p_1)^{1/2} \end{pmatrix} x' \begin{pmatrix} \sqrt{1-\lambda^2}1 & 0 \\ 0 & (1-\lambda^2 p_1^*p_1)^{1/2} \end{pmatrix},\end{aligned}$$

where

$$\begin{aligned} x' = &\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda^2}(1-\lambda^2 p_1^*p_1)^{-1/2}p_1^* \end{pmatrix} x_1 \begin{pmatrix} 1 & 0 \\ 0 & p_1\sqrt{1-\lambda^2}(1-\lambda^2 p_1^*p_1)^{-1/2} \end{pmatrix} \\ &+ \begin{pmatrix} 0 & 0 \\ 0 & (1-\lambda^2 p_1^*p_1)^{-1/2}p_2^* \end{pmatrix} x_2 \begin{pmatrix} 0 & 0 \\ 0 & p_2(1-\lambda^2 p_1^*p_1)^{-1/2} \end{pmatrix}.\end{aligned}$$

Observe that in order for x to be a proper C^*-convex combination of x_1 and x', it is sufficient that p_1 be invertible.

Technique C. If $\sum_{i=1} p_i^* p_i$ is invertible, then

$$\sum_i \begin{pmatrix} 0 & 0 \\ 0 & p_i^* \end{pmatrix} x_i \begin{pmatrix} 0 & 0 \\ 0 & p_i \end{pmatrix}$$

can be rewritten as

$$\begin{pmatrix} 0 & 0 \\ 0 & (\sum_j p_j^* p_j)^{1/2} \end{pmatrix} z \begin{pmatrix} 0 & 0 \\ 0 & (\sum_j p_j^* p_j)^{1/2} \end{pmatrix},$$

where $z \in C^*\text{-conv}\{x_i\}_i$ and is given by

$$z = \sum_i \begin{pmatrix} m^{-1/2}1 & 0 \\ 0 & (\sum_j p_j^* p_j)^{-1/2}p_i^* \end{pmatrix} x_i \begin{pmatrix} m^{-1/2}1 & 0 \\ 0 & p_i(\sum_j p_j^* p_j)^{-1/2} \end{pmatrix}.$$

Finally, we will make use of the following two elementary facts. One is that if $K \subset M_n$ is C^*-convex, and if $p \in M_n$ is a projection, then the compression pKp of K to the image of p is clearly C^*-convex. The other is that $x = x_1 \oplus x_2$

is C^*-extreme in K only if x_1 and x_2 are C^*-extreme in the corresponding compressed images of K. To prove the second fact, suppose that x_1, say, is not C^*-extreme in K_1, the corresponding compression of K. Then x_1 can be expressed as a proper C^*-convex combination $\sum_{i=1}^m s_i^* z_i s_i$ of elements $z_i \in K_1$, with at least one $z_i \not\sim x_1$. Thus,

$$x = \sum_{i=1}^{m} (s_i \oplus m^{-1/2}1)^*(z_i \oplus x_2)(s_i \oplus m^{-1/2}1)$$

is a representation of x as a proper C^*-convex combination of elements $z_i \oplus x_2 \in K$, where at least one $z_i \oplus x_2 \not\sim x_1 \oplus x_2$.

4.1. **Theorem.** *Suppose that $K = C^*\text{-conv}\{x_\alpha : \alpha \in I\} \subset M_n$, where I is any index set. If x is C^*-extreme in K, then x must be unitary equivalent to some x_α or reducible. Moreover, there exist projections q_i such that $\sum_i q_i = 1$, $x = \sum_i q_i x'_{\alpha_i} q_i$, and each $x'_{\alpha_i} \sim x_{\alpha_i}$.*

Proof. Let $x = \sum_{i=1}^m t_i^* x_{\alpha_i} t_i$ be a representation of x as a C^*-convex combination of the x_α using the fewest number of coefficients possible. (This least integer m does not change if x is replaced by some $x' \sim x$.) Because $m = 1$ if and only if $x \sim x_{\alpha_1}$, it is assumed henceforth that $m \geq 2$ and that x is not unitarily equivalent to some x_α. We are to prove that x is reducible.

We will assume without loss of generality that $\|t_1\| = 1$. Such an assumption is valid for the following reasons. If, on the contrary, $\|t_i\| < 1$ for every i, then x can be rewritten using Technique A as a proper C^*-convex combination of elements z_i. Because x is a C^*-extreme point of K, each z_i must be unitarily equivalent to x. However, the construction in Technique A shows that each z_i is a C^*-convex combination of fewer than m of the x_α; hence, x must possess this property of the z_i as well, in contradiction of the minimality of m. Thus, at least one coefficient t_i has unit norm.

By expressing each coefficient t_i in its polar decomposition $t_i = u_i a_i$, where u_i is unitary and $a_i \geq 0$, and by absorbing the unitary part u_i of t_i into x_{α_i}, we have that $x = \sum_i a_i x'_{\alpha_i} a_i$, where $a_i \geq 0$, $\sum_i a_i^2 = 1$, and $x'_{\alpha_i} = u_i^* x_{\alpha_i} u_i$. There is a unitary $u \in M_n$ for which $u^* a_1 u$ has a representation as a 2×2 operator matrix given by

$$u^* a_1 u = \begin{pmatrix} 1 & 0 \\ 0 & y_1 \end{pmatrix}$$

with $y_1 \geq 0$ and $\|y_1\| < 1$. (This follows from the fact that $a_1 \geq 0$ and $\|a_1\| = 1$.) Because $a_i \geq 0$ for all $i \geq 2$ and because $\sum_i a_i^2 = 1$, the same unitary u gives

$$u^* a_i u = \begin{pmatrix} 0 & 0 \\ 0 & y_i \end{pmatrix}$$

for $i \geq 2$.

Again, each $y_i \geq 0$. Hence,

$$x' = \begin{pmatrix} 1 & 0 \\ 0 & y_1 \end{pmatrix} x''_{\alpha_1} \begin{pmatrix} 1 & 0 \\ 0 & y_1 \end{pmatrix} + \sum_{i \geq 2} \begin{pmatrix} 0 & 0 \\ 0 & y_i \end{pmatrix} x''_{\alpha_i} \begin{pmatrix} 0 & 0 \\ 0 & y_i \end{pmatrix},$$

where $x' = uxu^*$ and $x''_{\alpha_i} = u^* x'_{\alpha_i} u$; for simplicity of notation, let us assume that

$$a_1 = \begin{pmatrix} 1 & 0 \\ 0 & y_1 \end{pmatrix}, \qquad a_i = \begin{pmatrix} 0 & 0 \\ 0 & y_i \end{pmatrix},$$

et cetera, so that $x = \sum_i a_i x_{\alpha_i} a_i$.

Now because $\|y_1\| < 1$, $1 - y_1^2$ is invertible; therefore, from

$$\sum_{i \geq 2} a_i^2 = \sum_{i \geq 2} \begin{pmatrix} 0 & 0 \\ 0 & y_i^2 \end{pmatrix} = 1 - a_1^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 - y_1^2 \end{pmatrix},$$

we conclude that $\sum_{i \geq 2} y_i^* y_i$ is invertible. Hence, Technique C allows us to write $\sum_{i \geq 2} a_i x_{\alpha_i} a_i$ as a single term $t_0^* x_0 t_0$, for some $x_0 \in K$. By passing to the polar decomposition of t_0 and absorbing the unitary part of t_0 into x_0, we may assume that $\sum_{i \geq 2} a_i x_{\alpha_i} a_i = a_0 x_0 a_0$ for some $a_0 \geq 0$ and $x_0 \in K$.

From $a_0^2 + a_1^2 = 1$, we see that

$$a_0 = \begin{pmatrix} 0 & 0 \\ 0 & y_0 \end{pmatrix}.$$

In fact, a_0 and a_1 admit a further decomposition so that x is given by

$$x = \begin{pmatrix} 1 & & \\ & p_1 & \\ & & 0 \end{pmatrix} x_{\alpha_1} \begin{pmatrix} 1 & & \\ & p_1 & \\ & & 0 \end{pmatrix} + \begin{pmatrix} 0 & & \\ & p_0 & \\ & & 1 \end{pmatrix} x_0 \begin{pmatrix} 0 & & \\ & p_0 & \\ & & 1 \end{pmatrix},$$

where $p_1 \geq 0$ is invertible and is of norm $\|p_1\| < 1$. We are now in a position to make repeated use of Technique B.

For each $\lambda \in (0, 1)$, we rewrite x (using Technique B) as

$$\begin{aligned} x = {} & \begin{pmatrix} \lambda 1 & & \\ & \lambda p_1 & \\ & & 0 \end{pmatrix} x_{\alpha_1} \begin{pmatrix} \lambda 1 & & \\ & \lambda p_1 & \\ & & 0 \end{pmatrix} \\ & + \begin{pmatrix} \sqrt{1-\lambda^2} 1 & & \\ & (1-\lambda^2 p_1^2)^{1/2} & \\ & & 1 \end{pmatrix} x(\lambda) \begin{pmatrix} \sqrt{1-\lambda^2} 1 & & \\ & (1-\lambda^2 p_1^2)^{1/2} & \\ & & 1 \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} x(\lambda) = {} & \begin{pmatrix} 1 & & \\ & \sqrt{1-\lambda^2}(1-\lambda^2 p_1^2)^{-1/2} p_1 & \\ & & 0 \end{pmatrix} x_{\alpha_1} \begin{pmatrix} 1 & & \\ & \sqrt{1-\lambda^2} p_1 (1-\lambda^2 p_1^2)^{1/2} & \\ & & 0 \end{pmatrix} \\ & + \begin{pmatrix} 0 & & \\ & (1-\lambda^2 p_1^2)^{-1/2} p_0 & \\ & & 1 \end{pmatrix} x_0 \begin{pmatrix} 0 & & \\ & p_0 (1-\lambda^2 p_1^2)^{-1/2} & \\ & & 1 \end{pmatrix}. \end{aligned}$$

In the expression for x, the (matrix) coefficients on $x(\lambda)$ are invertible for all $\lambda \in (0, 1)$. By grouping together the 2×2 upper left corner blocks of the coefficients in the expression for x, that is,

$$x = \begin{pmatrix} a(\lambda) & 0 \\ 0 & 0 \end{pmatrix} x_{\alpha_1} \begin{pmatrix} a(\lambda) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} b(\lambda) & 0 \\ 0 & 1 \end{pmatrix} x(\lambda) \begin{pmatrix} b(\lambda) & 0 \\ 0 & 1 \end{pmatrix},$$

we see that Technique B applies once again (though cosmetically modified since 1 appears in the bottom right corner rather than in the top left) to yield x as a proper C^*-convex combination

$$x = t(\lambda)^* x'' t(\lambda) + s(\lambda)^* x(\lambda) s(\lambda).$$

(This is a proper combination because $b(\lambda)$ is invertible for all $\lambda \in (0, 1)$.) Because x is C^*-extreme in K, $x(\lambda) \sim x$ for all $\lambda \in (0, 1)$. Moreover, because $x(\lambda)$ depends continuously on λ and because the unitary orbit of x is closed, $x(1) \sim x$. This proves that x is reducible, for $x(1)$ is given by

$$x(1) = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} x_{\alpha_1} \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} + \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix} x_0 \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix},$$

or equivalently, $x \sim x(1) = q_1 x_{\alpha_1} q_1 + q_1^\perp x_0 q_1^\perp$, where q_1 is a projection.

Recall that Technique C has x_0 as a C^*-convex combination of $x_{\alpha_2}, \dots, x_{\alpha_m}$ and so

$$x(1) = q_1 x_{\alpha_1} q_1 + \sum_{i \geq 2} s_i^* x_{\alpha_i} s_i,$$

where $q_1 + \sum_{i \geq 2} s_i^* s_i = 1$ and $q_1 \perp s_i$ for all $i \geq 2$. Observe that $\sum_{i \geq 2} s_i^* s_i = 1_{|_{\operatorname{Im} q_1^\perp}}$ and that the restriction of $x(1)$ to $\operatorname{Im} q_1^\perp$ is C^*-extreme in $q_1^\perp K q_1^\perp$ (by the second of our two elementary facts stated prior to the theorem); hence, we may repeat our earlier arguments to conclude that one of the s_i is of norm 1, say s_2, and to conclude further that there exists a projection q_2 such that

$$x \sim q_1 x_{\alpha_1} q_1 + q_2 x_{\alpha_2} q_2 + (\text{other terms}).$$

By exhausting this process after a finite number of steps, we see that $x \sim \sum_i q_i x_{\alpha_i} q_i$ for some projections q_i satisfying $\sum_i q_i = 1$. □

4.2. **Corollary.** *If $\mathscr{S} \subset M_n$ is a compact set of normals, then $K = C^*\text{-conv}\,\mathscr{S}$ has no nonnormal C^*-extreme points. Moreover, if $X \subset \mathbb{C}$ is the convex hull of the eigenvalues of the elements of $\mathscr{S}$, then $x \in K$ is C^*-extreme in K if and only if x is normal and the eigenvalues of x are extreme points of X.*

Proof. It is easy to see (e.g., [4]) that $K = C^*\text{-conv}\{\zeta 1 : \zeta$ is an extreme point of $X\}$. If $x \in K$ is C^*-extreme in K, then Theorem 4.1 shows that x is a scalar matrix $\zeta 1$, or is reducible and of the form $x = \sum_i q_i(\zeta_i 1) q_i$ for some projections q_i satisfying $\sum_i q_i = 1$. In either case, x is normal and the eigenvalues of x are extreme points of X. Conversely, every normal $x \in M_n$ with eigenvalues in the set of extreme points of X is C^*-extreme in K [4, Theorem 2]. □

In [4, Theorem 3] it was shown that the nilpotent Jordan matrix of order n is an extreme point of $K = C^*\text{-conv}\{\zeta 1_n : \zeta^{n+1} = 1\} \subset M_n$. Therefore, we now have a new set of examples, valid in each matrix algebra M_n, of C^*-convex sets having linear extreme points that are not C^*-extreme. (Previously, the only known example was in M_2 [5].)

The methods of this paper also allow us to recover all of the C^*-extreme points found by Hopenwasser, Moore, and Paulsen of the C^*-convex set in M_2 of matrices with numerical radius not exceeding 1. As is demonstrated by Arveson [1, p. 302], this set is generated by $\left(\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix}\right)$. In fact, we have the following general result.

4.3. **Theorem.** *If $x \in M_2$, then the C^*-extreme points of C^*-conv$\{x\}$ are all of the matrices unitarily equivalent to x and all of the scalar matrices obtained from the extreme points of the numerical range of x.*

Proof. If $x \in M_2$ is normal, then Corollary 4.2 yields the result. If $x \in M_2$ is nonnormal, then x is irreducible and so elements of the unitary orbit of x are C^*-extreme in C^*-conv$\{x\}$ (by Theorem 3.1), as are all scalar matrices $\zeta 1$ for which ζ is an extreme point of $W^1(x)$ [4, Theorem 1]. Conversely, suppose that z is C^*-extreme in C^*-conv$\{x\}$. If z is irreducible, then $z \sim x$; this follows from Theorem 4.1. If z is reducible, then it must be unitarily equivalent to a diagonal matrix, so let us say that

$$z = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$

If $\alpha \neq \beta$, then there exist linearly independent unit vectors $\xi, \eta \in \mathbb{C}^2$ such that $(z\xi, \xi) = \alpha$ and $(z\eta, \eta) = \beta$. The matrices

$$t_1 = \sqrt{2}^{-1}[\xi, -\eta] \in M_2, \qquad t_2 = \sqrt{2}^{-1}[\xi, \eta] \in M_2$$

are invertible and satisfy $t_1^* t_1 + t_2^* t_2 = 1$ and $t_1^* x t_1 + t_2^* x t_2 = z$. But because z is C^*-extreme in C^*-conv$\{x\}$, the (reducible) matrix z must be unitarily equivalent to the (irreducible) matrix x, thereby yielding a contradiction. Thus, z must be of the form $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha \end{smallmatrix}\right)$. Plainly, in order for the scalar matrix $\alpha 1_2$ to be extreme, let alone C^*-extreme, in $W^2(x)$, it is necessary that α be extreme in $W^1(x)$. □

5. Matricial ranges of a hyponormal operator

In their announcement [8], Pearcy and Salinas define the $n \times n$ reducing matricial spectrum of $x \in B(H)$ to be the (possibly void) set $R^n(x)$ of $n \times n$ matrices of the form $\rho(x)$, where $\rho : C^*(x) \to M_n$ is a unital $*$-homomorphism and $C^*(x)$ is the unital C^*-algebra generated by x. By [8, Theorem 2], $\lambda \in R^1(x)$ if and only if there exists a sequence of unit vectors $\xi_j \in H$ satisfying $\lim_j \|(x - \lambda 1)\xi_j\| = \lim_j \|(x - \lambda 1)^* \xi_j\| = 0$. Therefore, if x is hyponormal, meaning that $x^*x - xx^* \geq 0$, then $R^1(x)$ is precisely the approximate point spectrum $\sigma_l(x)$ of x, and it is $\sigma(x)$ if x is in fact normal. Thus, the well-known result $W^1(x) = \operatorname{conv} \sigma(x)$ for hyponormal x leads to $W^1(x) = \operatorname{conv} R^1(x)$ (by making use of the fact that the extreme points of $\operatorname{conv} \sigma(x)$ must be approximate eigenvalues). An interesting theorem of Stampfli goes even further: if $\lambda = (x\xi, \xi)$ is an extreme point of $W^1(x)$ for some unit vector $\xi \in H$, then $x\xi = \lambda\xi$ and $x^*\xi = \lambda^*\xi$ [10].

The purpose of this section is to show that these numerical range properties of hyponormal operators are present at the level of matricial ranges.

5.1. **Theorem.** *The following statements hold for every hyponormal operator x acting on a Hilbert space H.*

(1) $W^n(x) = C^*$-conv $R^n(x)$.
(2) *The C^*-extreme points of $W^n(x)$ are elements of $R^n(x)$.*
(3) *If $\Lambda = v^*xv$ is a C^*-extreme point of $W^n(x)$ for some isometry $v : \mathbb{C}^n \to H$, then vv^* is an invariant projection of x and x^*.*

Proof. (1) It is slightly more convenient to translate x by a scalar ζ so that the new hyponormal operator $\tilde{x} = x - \zeta 1$ has $0 \in \sigma_l(\tilde{x})$. Because the effect of translation on the matricial spectrum and the matricial range is just translation by the scalar matrix $\zeta 1_n$, we assume without loss of generality that already $0 \in \sigma_l(x)$. Because x is hyponormal, the dilation theorem [11] of Sz.-Nagy and Foiaş asserts that there exists a normal operator z on a Hilbert space H' and a contraction $a : H \to H'$ such that $x = a^* z a$ and $\sigma(z) \subset \sigma_l(x)$. The spectral inclusion as stated is in fact $R^1(z) \subset R^1(x)$, due to the normality of z and the hyponormality of x. If $R^1(z)$ does not contain 0, we can pass to the normal operator $z \oplus 0$ on $H' \oplus \mathbb{C}$, having 0 in its spectrum, so that x is now of the form $a_0^*(z \oplus 0)a_0$, where a_0 is the contraction obtained by the composition $a_0 = pqa$ with q denoting the canonical injection $H' \to H' \oplus \mathbb{C}$ and p denoting the canonical projection $H' \oplus \mathbb{C} \to H'$. Hence, again without loss of generality, we will assume that already $0 \in R^1(z)$.

We first show that $W^n(x) \subset W^n(z)$. Because $0 \in (R^1(z) \otimes 1_n)$, it follows that $\lambda^* W^n(z) \lambda \subset W^n(z)$ for every contraction $\lambda \in M_n$, by Remark 7 of [6]. Suppose now that ϕ is a unital completely positive map $C^*(x) \to M_n$; we will show that $\phi(x) \in W^n(z)$. The argument runs similar to that of Lemma 4.2 in [9]. Consider the completely positive map $\psi : C^*(z) \to M_n$ defined by $\psi(f) = \phi(a^* f a)$, with $\lambda = \psi(1)$. For the positive contraction λ there exists $b \geq 0$ in M_n such that if p is the projection of $\mathbb{C}^n$ onto the range of λ, then $b\lambda b = p$ (we have $b = \lambda^{-1/2}$ on the range of p). Choose any state θ on $C^*(z)$ and consider the unital completely positive map $\omega : C^*(z) \to M_n$ defined by $\omega(f) = b\psi(f)b + \theta(f)(1-p)$. From $\lambda^{1/2}\omega(z)\lambda^{1/2} = \phi(x)$ it follows that

$$\phi(x) \in \lambda^{1/2} W^n(z) \lambda^{1/2} \subset W^n(z),$$

as desired.

Because the sets $R^n(x)$ and $R^n(z)$ consist solely of normal matrices, their C^*-convex hulls are determined by their eigenvalue sets; that is, $R^1(x)$ and $R^1(z)$ determine the C^*-convex hulls of $R^n(x)$ and $R^n(z)$, respectively. Hence,

$$W^n(z) = C^*\text{-conv}\, R^1(z) \subset C^*\text{-conv}\, R^1(x) \subset W^n(x) \subset W^n(z)\,.$$

(2) By (1), $W^n(x)$ is the C^*-convex hull of a compact set of normals; hence, if $\Lambda \in W^n(x)$ is a C^*-extreme point of $W^n(x)$, then Corollary 4.2 has that Λ must be normal and its eigenvalues must be extreme in the convex hull of the spectra of $R^n(x)$ (viz., the eigenvalues of Λ are extreme points of $\operatorname{conv} R^1(x) = W^1(x)$). Each eigenvalue of Λ determines a unital $*$-homomorphism $C^*(x) \to \mathbb{C}$ and by passing to a direct sum of these homomorphisms, we obtain a unital $*$-homomorphism $\rho : C^*(x) \to M_n$ such that $\rho(x)$ is unitarily equivalent to Λ.

(3) By (2), the C^*-extreme point $\Lambda = v^* x v$ is normal and $\sigma(\Lambda)$ is a subset of the extreme points of $W^1(x)$. Thus, whenever ξ is a unit eigenvector of Λ with corresponding eigenvalue λ, then $\lambda = (xv\xi, v\xi)$ is an extreme point of $W^1(x)$. By Stampfli's theorem, $x(v\xi) = \lambda(v\xi)$ and $x^*(v\xi) = \lambda^*(v\xi)$. Furthermore, as Λ is completely determined by its spectral structure, we conclude that $xv = v\Lambda$ and $x^* v = v\Lambda^*$, whence vv^* commutes with x. $\square$

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