A SIMPLE FORMULA FOR CYCLIC DUALITY

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Abstract. We give a simple formula for duality in an easily described covering category of the cyclic category and show that the formula given descends to the cyclic category.

The cyclic category, $\Lambda$, was introduced by Connes in [1] as part of his project on noncommutative differential geometry. Since then it has been studied extensively by topologists, especially those interested in the algebraic $K$-theory of spaces; see, for instance, [2–6]. One of its crucial features is self-duality: $\Lambda \cong \Lambda^{\text{op}}$, where $\Lambda^{\text{op}}$ is the opposite category of $\Lambda$. This duality is usually described, in a fairly unilluminating fashion, in terms of its effect on generators. (There are actually infinitely many such dualities.) The purpose of this note is to give a simple formula for one such duality (in fact, the one originally given by Connes in [1]) in terms of an elementary combinatorial model for $\Lambda$; the formula makes it quite easy to describe the cyclic structure on topological Hochschild homology for an arbitrary $A_\infty$ ring spectrum, as will be shown elsewhere.

We model $\Lambda$ as a quotient of a category that is of interest in its own right, which we call the linear category $L$. The objects of $L$ are the nonnegative integers $\{0, 1, 2, \ldots\}$, each thought of as a separate copy of $\mathbb{Z}$. The morphisms are functions from $\mathbb{Z}$ to $\mathbb{Z}$; for $f \in L(m, n)$, we require

1. $i_1 \leq i_2 \Rightarrow f(i_1) \leq f(i_2)$, and
2. $f(i + m + 1) = f(i) + n + 1$ for all $i$.

Composition in $L$ is composition as functions.

In terms of generators and relations, we will see that $L$ has the face and degeneracy operators of $\Lambda$, the simplicial category, and generators $\gamma_n \in L(n, n)$ given by $\gamma_n(i) = i + 1$ that map to the cyclic generators $\tau_n \in \Lambda(n, n)$. However, the relation from $\Lambda$ that $\tau_{n+1}^n = 1$ is dropped for $\gamma_n$, although we retain the property that $\gamma_n$ is invertible; this last requirement distinguishes $L$ from the duplicial category of Dwyer and Kan [3].

The following proposition gives the duality formula in $L$; we will then show that it descends to $\Lambda$.

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709
Proposition 1. The following formula gives an isomorphism $D : L \cong L^{\text{op}}$ that is the identity on objects:

$$Df(j) \leq i \iff j \leq f(i).$$

Proof. Noting that the formula is equivalent to requiring

$$Df(j) > i \iff j > f(i),$$

it is easy to see that $D$ is a contravariant functor with inverse $D^{-1}$ given by

$$g(j) \leq i \iff j \leq D^{-1}g(i).$$

We next identify the generators and relations in $L$, in order to identify $A$ as a quotient of $L$.

Lemma 2. For any $f$ in $L$, $f \circ Df \circ f = f$. Consequently, $Df$ is a right inverse for $f$ if it has one, and also a left inverse for $f$ if it has one.

Proof. We have $Df(j) \leq Df(j) \Rightarrow j \leq f(Df(j))$, so by replacement, $f(i) \leq f(Df(f(i)))$. On the other hand, $f(i) \leq f(i) \Rightarrow Df(f(i)) \leq i$, so applying $f$, we see that $f(Df(f(i))) \leq f(i)$. The conclusion follows. □

Lemma 3. The simplicial category $\Delta$ embeds in $L$, with $f \in \Delta(m, n)$ if and only if $0 \leq f(0)$ and $f(m) \leq n$.

Proof. Given $f \in \Delta(m, n)$ as a nondecreasing function from $\{0, \ldots, m\}$ to $\{0, \ldots, n\}$, there is a unique morphism $f' \in L(m, n)$ with $f'(i) = f(i)$ for $0 \leq i \leq m$. (Uniqueness follows from property (2) of the definition of $L$.) We identify $f$ with $f'$, and clearly $0 \leq f(0)$ and $f(m) \leq n$. Conversely, any such $f$ determines an element of $\Delta(m, n)$. □

Proposition 4. Let $L_0$ be the subcategory of $L$ given by $L_0 = \{f : f(0) = 0\}$. Then $D(\Delta) = L_0$, and consequently $L_0 \cong \Delta^{\text{op}}$.

Proof. Using Lemma 3, the fact that $f(m) \leq n \iff f(m) < n + 1$, and property (2) of the definition of $L$, we see that

$$f \in \Delta \iff f(-1) < 0 \leq f(0)$$

$$\iff -1 < Df(0) \leq 0$$

$$\iff Df(0) = 0.$$ □

Next we need the canonical morphisms $\gamma_n \in L(n, n)$ given by $\gamma_n(i) = i + 1$ and their inverses $\beta_n(i) = i - 1$. By Lemma 2, $D\gamma_n = \beta_n$.

Lemma 5. Every morphism $f$ in $L$ factors uniquely as $\gamma^k \circ g$ with $g \in L_0$; the integer $k$ must be $f(0)$. There is a natural action of $\mathbb{Z}$ on $L_0(m, n)$; writing the action of $k$ on $g$ as $g_k$, it is given by the formula

$$g_k(i) = g(i + k) - g(k).$$

We then have $g \circ \gamma^k = \gamma^{g(k)} \circ g_k$. Notice that the $\mathbb{Z}$-action actually factors through $\mathbb{Z}/m + 1$.

Proof. This is completely trivial. □

Corollary 6. Every morphism $f$ in $L$ factors uniquely as $\phi \circ \beta^k$ for $\phi \in \Delta$. There is a natural $\mathbb{Z}$-action on $\Delta(m, n)$ defined by $D(\phi_k) = (D\phi)_k$ that consequently factors through $\mathbb{Z}/n + 1$; we have the commutation formula

$$\beta^k \circ \phi = \phi_k \circ \beta^{D\phi(k)}.$$
Proof. Factor $Df$ as $\gamma^k \circ g$; then $f = D^{-1}g \circ \beta^k$, so $\phi = D^{-1}g$. For the commutation formula, take duals:

$$D(\beta^k \circ \phi) = D\phi \circ \gamma^k = \gamma^{D\phi(k)} \circ D\phi_k$$

$$\Rightarrow \beta^k \circ \phi = \phi_k \circ \beta^{D\phi(k)}.$$  □

**Corollary 7.** The cyclic category $\Lambda$ is the quotient category of $L$ obtained by setting $\beta^m_n = \text{id}$. Further, the isomorphism $D$ descends to an isomorphism $\Lambda \cong \Lambda^{\text{op}}$.

Proof. A morphism in $L(m,n)$ induces a map $\mathbb{Z}/m + 1 \to \mathbb{Z}/n + 1$, and since the action in Corollary 6 of $\mathbb{Z}$ on $\Delta(m,n)$ factors through $\mathbb{Z}/n + 1$, the commutation formula makes sense with $\beta_m$ as a generator of $\mathbb{Z}/m + 1$. The usual commutation formulas, e.g., of [5], now follow easily. To see that $D$ descends to $\Lambda$, note that $f \sim g$ if and only if $f = g \circ \beta^{k(m+1)} = \beta^{k(n+1)} \circ g$ for some $k$. But then $Df = \beta^{-k(m+1)} \circ Dg = Dg \circ \beta^{-k(n+1)}$, so $D$ respects the quotient map. □

In conclusion, we remark that since the linear category $L$ contains a copy of $\Lambda$, it makes sense to talk about the geometric realization of a “linear space,” and it is not hard to see that such a geometric realization has a natural action by the additive group of real numbers that descends to the usual action of the circle group if the functor factors through the cyclic category.

**References**


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