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A SIMPLE FORMULA FOR CYCLIC DUALITY

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ABSTRACT. We give a simple formula for duality in an easily described covering category of the cyclic category and show that the formula given descends to the cyclic category.

The cyclic category, Λ, was introduced by Connes in [1] as part of his project on noncommutative differential geometry. Since then it has been studied extensively by topologists, especially those interested in the algebraic K-theory of spaces; see, for instance, [2–6]. One of its crucial features is self-duality: $\Lambda \cong \Lambda^{\mathrm{op}}$, where Λ^{op} is the opposite category of Λ. This duality is usually described, in a fairly unilluminating fashion, in terms of its effect on generators. (There are actually infinitely many such dualities.) The purpose of this note is to give a simple formula for one such duality (in fact, the one originally given by Connes in [1]) in terms of an elementary combinatorial model for Λ; the formula makes it quite easy to describe the cyclic structure on topological Hochschild homology for an arbitrary A_∞ ring spectrum, as will be shown elsewhere.

We model Λ as a quotient of a category that is of interest in its own right, which we call the *linear* category L. The objects of L are the nonnegative integers $\{0, 1, 2, \dots\}$, each thought of as a separate copy of $\mathbb{Z}$. The morphisms are functions from $\mathbb{Z}$ to $\mathbb{Z}$; for $f \in L(m, n)$, we require

(1) $i_1 \le i_2 \Rightarrow f(i_1) \le f(i_2)$, and
(2) $f(i+m+1) = f(i) + n + 1$ for all i.

Composition in L is composition as functions.

In terms of generators and relations, we will see that L has the face and degeneracy operators of Δ, the simplicial category, and generators $\gamma_n \in L(n, n)$ given by $\gamma_n(i) = i+1$ that map to the cyclic generators $\tau_n \in \Lambda(n, n)$. However, the relation from Λ that $\tau_n^{n+1} = 1$ is dropped for γ_n, although we retain the property that γ_n is invertible; this last requirement distinguishes L from the duplicial category of Dwyer and Kan [3].

The following proposition gives the duality formula in L; we will then show that it descends to Λ.

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Proposition 1. *The following formula gives an isomorphism* $D : L \cong L^{\mathrm{op}}$ *that is the identity on objects*:

$$Df(j) \leq i \Leftrightarrow j \leq f(i).$$

Proof. Noting that the formula is equivalent to requiring

$$Df(j) > i \Leftrightarrow j > f(i),$$

it is easy to see that D is a contravariant functor with inverse D^{-1} given by

$$g(j) \leq i \Leftrightarrow j \leq D^{-1}g(i). \quad \square$$

We next identify the generators and relations in L, in order to identify Λ as a quotient of L.

Lemma 2. *For any* f *in* L, $f \circ Df \circ f = f$. *Consequently,* Df *is a right inverse for* f *if it has one, and also a left inverse for* f *if it has one.*

Proof. We have $Df(j) \leq Df(j) \Rightarrow j \leq f(Df(j))$, so by replacement, $f(i) \leq f(Df(f(i)))$. On the other hand, $f(i) \leq f(i) \Rightarrow Df(f(i)) \leq i$, so applying f, we see that $f(Df(f(i))) \leq f(i)$. The conclusion follows. $\square$

Lemma 3. *The simplicial category* Δ *embeds in* L, *with* $f \in \Delta(m, n)$ *if and only if* $0 \leq f(0)$ *and* $f(m) \leq n$.

Proof. Given $f \in \Delta(m, n)$ as a nondecreasing function from $\{0, \dots, m\}$ to $\{0, \dots, n\}$, there is a unique morphism $f' \in L(m, n)$ with $f'(i) = f(i)$ for $0 \leq i \leq m$. (Uniqueness follows from property (2) of the definition of L.) We identify f with f', and clearly $0 \leq f(0)$ and $f(m) \leq n$. Conversely, any such f determines an element of $\Delta(m, n)$. $\square$

Proposition 4. *Let* L_0 *be the subcategory of* L *given by* $L_0 = \{f : f(0) = 0\}$. *Then* $D(\Delta) = L_0$, *and consequently* $L_0 \cong \Delta^{\mathrm{op}}$.

Proof. Using Lemma 3, the fact that $f(m) \leq n \Leftrightarrow f(m) < n + 1$, and property (2) of the definition of L, we see that

$$\begin{aligned} f \in \Delta &\Leftrightarrow f(-1) < 0 \leq f(0) \\ &\Leftrightarrow -1 < Df(0) \leq 0 \\ &\Leftrightarrow Df(0) = 0. \quad \square \end{aligned}$$

Next we need the canonical morphisms $\gamma_n \in L(n, n)$ given by $\gamma_n(i) = i + 1$ and their inverses $\beta_n(i) = i - 1$. By Lemma 2, $D\gamma_n = \beta_n$.

Lemma 5. *Every morphism* f *in* L *factors uniquely as* $\gamma^k \circ g$ *with* $g \in L_0$; *the integer* k *must be* $f(0)$. *There is a natural action of* $\mathbb{Z}$ *on* $L_0(m, n)$; *writing the action of* k *on* g *as* g_k, *it is given by the formula*

$$g_k(i) = g(i + k) - g(k).$$

We then have $g \circ \gamma^k = \gamma^{g(k)} \circ g_k$. Notice that the $\mathbb{Z}$-action actually factors through $\mathbb{Z}/m + 1$.

Proof. This is completely trivial. $\square$

Corollary 6. *Every morphism* f *in* L *factors uniquely as* $\phi \circ \beta^k$ *for* $\phi \in \Delta$. *There is a natural* $\mathbb{Z}$*-action on* $\Delta(m, n)$ *defined by* $D(\phi_k) = (D\phi)_k$ *that consequently factors through* $\mathbb{Z}/n + 1$; *we have the commutation formula*

$$\beta^k \circ \phi = \phi_k \circ \beta^{D\phi(k)}.$$

Proof. Factor Df as $\gamma^k \circ g$; then $f = D^{-1}g \circ \beta^k$, so $\phi = D^{-1}g$. For the commutation formula, take duals:

$$\begin{aligned} D(\beta^k \circ \phi) &= D\phi \circ \gamma^k = \gamma^{D\phi(k)} \circ D\phi_k \\ &\Rightarrow \beta^k \circ \phi = \phi_k \circ \beta^{D\phi(k)}. \quad \square \end{aligned}$$

Corollary 7. *The cyclic category* Λ *is the quotient category of* L *obtained by setting* $\beta_n^{n+1} = \mathrm{id}$. *Further, the isomorphism* D *descends to an isomorphism* $\Lambda \cong \Lambda^{\mathrm{op}}$.

Proof. A morphism in $L(m, n)$ induces a map $\mathbb{Z}/m+1 \to \mathbb{Z}/n+1$, and since the action in Corollary 6 of $\mathbb{Z}$ on $\Delta(m, n)$ factors through $\mathbb{Z}/n+1$, the commutation formula makes sense with β_m as a generator of $\mathbb{Z}/m+1$. The usual commutation formulas, e.g., of [5], now follow easily. To see that D descends to Λ, note that $f \sim g$ if and only if $f = g \circ \beta^{k(m+1)} = \beta^{k(n+1)} \circ g$ for some k. But then $Df = \beta^{-k(m+1)} \circ Dg = Dg \circ \beta^{-k(n+1)}$, so D respects the quotient map. $\square$

In conclusion, we remark that since the linear category L contains a copy of Δ, it makes sense to talk about the geometric realization of a "linear space," and it is not hard to see that such a geometric realization has a natural action by the additive group of real numbers that descends to the usual action of the circle group if the functor factors through the cyclic category.

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