

A SHARP BOUND FOR THE REGULARITY INDEX OF FAT POINTS IN GENERAL POSITION

M. V. CATALISANO, N. V. TRUNG, AND G. VALLA

(Communicated by Louis J. Ratliff, Jr.)

ABSTRACT. A bound is given for the regularity index of the coordinate ring of a set of fat points in general position in \mathbf{P}_k^n . The bound is attained by points on a rational normal curve.

INTRODUCTION

Let P_1, \dots, P_s be distinct points in \mathbf{P}_k^n , k an algebraically closed field, and let m_1, \dots, m_s be positive integers. If \wp_1, \dots, \wp_s are the prime ideals in $R := k[X_0, \dots, X_n]$ corresponding to the points P_1, \dots, P_s , we let $Z := m_1 P_1 + \dots + m_s P_s$ be the zero cycle defined by the ideal $\wp_1^{m_1} \cap \wp_2^{m_2} \cap \dots \cap \wp_s^{m_s}$. If $m_i \geq 2$ the point P_i is called a *fat point* of Z , a self-explanatory term. There is some interest in calculating the Hilbert function and the Betti numbers of the graded ring

$$A = R/(\wp_1^{m_1} \cap \wp_2^{m_2} \cap \dots \cap \wp_s^{m_s}),$$

which is the homogeneous coordinate ring of Z .

It is well known that $A = \bigoplus_{t \geq 0} A_t$ is a one-dimensional Cohen-Macaulay graded ring whose multiplicity is $e := \sum_{i=1}^s \binom{m_i+n-1}{n}$, the degree of Z . This implies that the Hilbert function $H_A(t) := \dim_k A_t$ of A is strictly increasing until it reaches the multiplicity, at which it stabilizes. The least integer t for which $H_A(t) = e$ is called the *regularity index* of A and denoted by $r(A)$. Note that since A is Cohen-Macaulay, if L is any linear form not vanishing at any of the points P_1, \dots, P_s , then the artinian ring $B = A/LA$ has the property that $B_t = 0$ iff $t > r(A)$. So, for an artinian ring B we shall call its regularity the least integer t such that $H_B(t) = 0$.

Different results have been given on the postulation of the scheme Z in the case of fat points (see [C, DG, C1, C2, G, G1, G2, H, S]), but they have been proved mostly for $n = 2$; the papers [Hi] and [A] by Hirschowitz and Alexander, respectively, cover most of the known results for $n \geq 3$.

In this paper, by using elementary linear algebra, we give an upper bound for the regularity index of A when the points are in general position (see Theorem

Received by the editors February 11, 1991 and, in revised form, November 6, 1991.

1991 *Mathematics Subject Classification.* Primary 14M05, 14C20; Secondary 13H15, 13D40.

Key words and phrases. Regularity index, points in general position, rational normal curve.

The first and third authors were partially supported by M.P.I. (Italy).

6). This bound is attained for points lying on a rational normal curve (see Proposition 7). When $n = 2$ we get the same bound found in [C1] (see also [CG, Theorem 3.3]). Our approach is purely algebraic but in the last part of the paper we also give a geometric interpretation of the proofs.

MAIN RESULT

In this section we state and prove the main result of the paper, Proposition 5.

Many of the proofs require induction arguments on the number of points being considered. We begin by finding the index of regularity of the ring $A = k[X_0, \dots, X_n]/\wp^a$, where (with no loss of generality) we assume $\wp = (X_1, \dots, X_n)$ corresponds to the point $P = (1, 0, \dots, 0)$. Since $X_0 \notin \wp$ we can use our observation above on the ring $B = A/X_0A$ to see that $r(A) = a - 1$.

Lemma 1. *Let P_1, \dots, P_r, P be distinct points in \mathbf{P}_k^n and let \wp be the defining prime ideal of P . If m_1, \dots, m_r , and a are positive integers, $J := \wp_1^{m_1} \cap \wp_2^{m_2} \cap \dots \cap \wp_r^{m_r}$, and $I := J \cap \wp^a$, then*

$$r(R/I) = \max\{a - 1, r(R/J), r(R/(J + \wp^a))\}.$$

Proof. From the exact sequence of vector spaces

$$0 \rightarrow I_t \rightarrow J_t \oplus (\wp^a)_t \rightarrow (J + \wp^a)_t \rightarrow 0$$

it is clear that

$$H_{R/I}(t) = H_{R/\wp^a}(t) + H_{R/J}(t) - H_{R/(J+\wp^a)}(t)$$

for every integer t . Since the Hilbert functions of the one-dimensional Cohen-Macaulay rings R/I , R/\wp^a , and R/J are strictly increasing until they reach the multiplicity of the ring in question and since $R/(J + \wp^a)$ is artinian, we see that $H_{R/I}(t) = e(R/I)$ if and only if $H_{R/\wp^a}(t) = a - 1 = e(R/\wp^a)$, $H_{R/J}(t) = e(R/J)$, and $H_{R/(J+\wp^a)}(t) = 0$. The conclusion follows.

As a consequence of the above result we get the following lower bound for the regularity index of any zero-cycle in \mathbf{P}_k^n .

Corollary 2. *Let $s \geq 2$, P_1, \dots, P_s be distinct points in \mathbf{P}_k^n , and $m_1 \geq m_2 \geq \dots \geq m_s$ be positive integers. If $I := \wp_1^{m_1} \cap \wp_2^{m_2} \cap \dots \cap \wp_s^{m_s}$ then $r(R/I) \geq m_1 + m_2 - 1$.*

Proof. We may assume $\wp_1 = (X_0, X_2, \dots, X_n)$ and $\wp_2 = (X_1, \dots, X_n)$, and we let $J = \wp_1^{m_1} \cap \wp_3^{m_3} \cap \dots \cap \wp_s^{m_s}$. Then it is clear that $X_0^{m_1-1} X_1^{m_2-1} \notin \wp_1^{m_1} + \wp_2^{m_2}$. Since $J + \wp_2^{m_2} \subseteq \wp_1^{m_1} + \wp_2^{m_2}$, it follows that $X_0^{m_1-1} X_1^{m_2-1} \notin J + \wp_2^{m_2}$. This proves that $r(R/(J + \wp_2^{m_2})) \geq m_1 + m_2 - 1$ and the conclusion follows by the above lemma.

Remark. If we fix the exponents m_1, \dots, m_s , then the best lower bound for $r(R/I)$ is given by the regularity index of s generic fat points with those exponents. This bound is difficult to compute (see the papers [Hi, A] for the special case $m_1 = \dots = m_s = 2$).

If we want to use the formula given in Lemma 1 then we need to find a good bound for the regularity index of the graded ring $R/(J + \wp^a)$. In the following lemma we give some basic properties of this artinian ring.

Lemma 3. Let P_1, \dots, P_r, P be distinct points in \mathbf{P}_k^n and \wp the defining prime ideal of P . If m_1, \dots, m_r, a are positive integers and $J := \wp_1^{m_1} \cap \wp_2^{m_2} \cap \dots \cap \wp_r^{m_r}$, then

(a) $H_{R/(J+\wp^a)}(t) = \sum_{i=0}^{a-1} \dim_k [(J + \wp^i)/(J + \wp^{i+1})]_t$ for every $t \geq 0$.

(b) If $P = (1, 0, \dots, 0)$ then $[(J + \wp^i)/(J + \wp^{i+1})]_t = 0$ if and only if $i > t$ or $X_0^{t-i}M \in J + \wp^{i+1}$ for every monomial M of degree i in X_1, \dots, X_n .

Proof. The first assertion follows by using the exact sequences

$$0 \rightarrow (J + \wp^i)/(J + \wp^{i+1}) \rightarrow R/(J + \wp^{i+1}) \rightarrow R/(J + \wp^i) \rightarrow 0$$

for $i = 1, \dots, a - 1$. As for (b) it follows easily from the assumption $P = (1, 0, \dots, 0)$ which implies $\wp = (X_1, \dots, X_n)$.

Now we need the following result, which has a combinatorial flavour. Recall that a set of points in \mathbf{P}_k^n is said to be in *general position* if no $h + 2$ of them are on an h -plane for $h < n$.

Lemma 4. Let P_1, \dots, P_r, P be distinct points in general position in \mathbf{P}_k^n , let $m_1 \geq \dots \geq m_r$ be positive integers, and let $J := \wp_1^{m_1} \cap \wp_2^{m_2} \cap \dots \cap \wp_r^{m_r}$. If t is an integer such that $nt \geq \sum m_i$ and $t \geq m_1$, we can find t hyperplanes, say L_1, \dots, L_t , avoiding P such that $L_1 \cdots L_t \in J$.

Proof. If $r \leq n$, by the general position assumption we can find a hyperplane L avoiding P and passing through P_1, \dots, P_r . Since $t \geq m_1$, we have

$$L^t \in \wp_1^t \cap \wp_2^t \cap \dots \cap \wp_r^t \subseteq J.$$

Hence we get the conclusion if $r \leq n$ and, in particular, if $\sum m_i \leq n$. Thus we may assume $r \geq n + 1$ and argue by induction on $\sum m_i$. By the general position assumption we may find a hyperplane, say L , avoiding P and passing through P_1, \dots, P_n . Since $nt \geq \sum m_i$, we have

$$n(t - 1) \geq \sum m_i - n = (m_1 - 1) + \dots + (m_n - 1) + m_{n+1} + \dots + m_r.$$

On the other hand, since $t \geq m_1$ and $nt \geq \sum m_i \geq (n + 1)m_{n+1}$, it follows that

$$t - 1 \geq \{m_1 - 1, \dots, m_n - 1, m_{n+1}, \dots, m_r\}.$$

Thus, by the inductive assumption we can find $t - 1$ hyperplanes, say L_2, \dots, L_t avoiding P and such that

$$L_2 \cdots L_t \in \wp_1^{m_1-1} \cap \dots \cap \wp_n^{m_n-1} \cap \wp_{n+1}^{m_{n+1}} \cap \dots \cap \wp_r^{m_r}.$$

This implies $LL_2 \cdots L_t \in J$ and the conclusion follows.

Now we come to the main result of this paper.

Proposition 5. Let P_1, \dots, P_r, P be distinct points in general position in \mathbf{P}_k^n and \wp the defining prime ideal of P . Further let $m_1 \geq \dots \geq m_r \geq a$ be positive integers, $J := \wp_1^{m_1} \cap \wp_2^{m_2} \cap \dots \cap \wp_r^{m_r}$, $m := \sum m_i$, and t the least integer such that $nt \geq m + a - 1$. Then

$$r(R/(J + \wp^a)) \leq \max\{m_1 + a - 1, t\}.$$

Proof. Let us assume $P = (1, 0, \dots, 0)$ and so $\wp = (X_1, \dots, X_n)$. If $r \leq n$ we can find a hyperplane, say L , such that L contains P_1, \dots, P_r and does not contain P . Then, by scaling if necessary, $L = X_0 + H$ for some linear form

$H \in \wp$. Since it is clear that $L^{m_1} \in J$, we get $X_0^{m_1} \in J + \wp$. Let i be any integer $0 \leq i \leq a - 1$, and let M be a monomial of degree i in X_1, \dots, X_n . Then $M \in \wp^i$, hence $X_0^{m_1} M \in J + \wp^{i+1}$, which implies $X_0^{m_1+a-1-i} M \in J + \wp^{i+1}$. By Lemma 3 we get $r(R/(J + \wp^a)) \leq m_1 + a - 1$, as desired.

Now let $r \geq n$; after a suitable change of coordinates we may further assume that $P_1 = (0, 1, 0, \dots, 0), \dots, P_n = (0, 0, \dots, 1)$.

Let $h = \max\{m_1 + a - 1, t\}$. By Lemma 3(a) we need to prove that

$$[(J + \wp^i)/(J + \wp^{i+1})]_h = 0$$

for every $i = 0, \dots, a - 1$. Since $h > i$, by Lemma 3(b) this is equivalent to proving that

$$X_0^{h-i} M \in J + \wp^{i+1}$$

for every $i = 0, \dots, a - 1$ and every monomial M of degree i in X_1, \dots, X_n . Let $M + X_1^{c_1} \dots X_n^{c_n}$ with $\sum c_k = i$. Then $m_j + c_j \geq m_j \geq a > a - 1 \geq i$, hence $m_j - i + c_j > 0$.

On the other hand, for every $j = 1, \dots, n$, we have $c_j \leq i$, hence $m_j - i + c_j \leq m_j$. Since $h \geq m_1 + a - 1$ and $i \leq a - 1$, this implies

$$h - i \geq m_1 \geq \max\{m_1 - i + c_1, \dots, m_n - i + c_n, m_{n+1}, \dots, m_r\}.$$

Moreover, $h \geq t$, hence $nh \geq m + a - 1$. This implies

$$n(h - i) \geq m + a - 1 - ni \geq m + i - ni = \sum_{j=1}^n (m_j - i + c_j) + \sum_{j=n+1}^r m_j.$$

Thus we may use Lemma 4 to find $h - i$ hyperplanes, say F_1, \dots, F_{h-i} , avoiding P and such that

$$F_1 \dots F_{h-i} \in \wp_1^{m_1-i+c_1} \cap \dots \cap \wp_n^{m_n-i+c_n} \cap \wp_{n+1}^{m_{n+1}} \cap \dots \cap \wp_r^{m_r}.$$

Since it is clear that

$$M = X_1^{c_1} \dots X_n^{c_n} \in \wp_1^{i-c_1} \cap \dots \cap \wp_n^{i-c_n},$$

we get $MF_1 \dots F_{h-i} \in J$.

But the hyperplanes F_j do not contain P , hence for every $j = 1, \dots, h - i$ we can write $F_j = X_0 + G_j$ for suitable linear forms $G_j \in \wp$. We get $M(X_0 + G_1) \dots (X_0 + G_{h-i}) \in J$ and since $MG_j \in \wp^{i+1}$ for every j , this implies $X_0^{h-i} M \in J + \wp^{i+1}$ as wanted.

From now on if α is a rational number, denote by $[\alpha]$ its integer part. With this notation it is clear that for positive integers q and n we have

$$q + n > n[(q + n - 1)/n] \geq q.$$

This implies

$$[(q + n - 1)/n] = \min\{t | tn \geq q\}.$$

Theorem 6. Let $s \geq 2$, P_1, \dots, P_s be distinct points in general position in \mathbf{P}_k^n , and $m_1 \geq \dots \geq m_s$ be positive integers. Further let $I = \wp_1^{m_1} \cap \dots \cap \wp_s^{m_s}$. Then

$$r(R/I) \leq \max\left\{m_1 + m_2 - 1, \left[\left(\sum m_i + n - 2\right)/n\right]\right\}.$$

Proof. Let $J := \wp_1^{m_1} \cap \dots \cap \wp_{s-1}^{m_{s-1}}$. By Lemma 1 we have

$$r(R/I) = \max\{m_s - 1, r(R/J), r(R/(J + \wp_s^{m_s}))\}.$$

Let $s = 2$. Since $m_1 + m_2 - 1 \geq \min\{t \mid nt \geq m_1 + m_2 - 1\}$, we get

$$r(R/(J + \wp_2^{m_2})) \leq m_1 + m_2 - 1$$

by Proposition 5. Note that $r(R/J) = r(R/\wp_1^{m_1}) = m_1 - 1$. Then $r(R/I) \leq m_1 + m_2 - 1$ and the conclusion follows in this case. Thus we may argue by induction on s . By the inductive assumption we have

$$r(R/J) \leq \max \left\{ m_1 + m_2 - 1, \left[\left(\sum_{i=1}^{s-1} m_i + n - 2 \right) / n \right] \right\},$$

and by the above proposition

$$r(R/(J + \wp_s^{m_s})) \leq \max \left\{ m_1 + m_s - 1, \left[\left(\sum_{i=1}^s m_i + n - 2 \right) / n \right] \right\}.$$

Hence the conclusion is immediate.

We prove now that the bound found in Theorem 6 is sharp for points lying on a rational normal curve.

Proposition 7. *Let $s \geq 2$, P_1, \dots, P_s be distinct points on a rational normal curve in \mathbf{P}_k^n , and $m_1 \geq \dots \geq m_s$ be positive integers. Further let $I = \wp_1^{m_1} \cap \dots \cap \wp_s^{m_s}$. Then*

$$r(R/I) = \max \left\{ m_1 + m_2 - 1, \left[\left(\sum m_i + n - 2 \right) / n \right] \right\}.$$

Proof. We recall that all rational normal curves in \mathbf{P}^n are isomorphic under a linear change of coordinates. Hence, without loss of generality, we may assume that the points are on the curve with parametric equations

$$X_0 = t^n, \quad X_1 = t^{n-1}u, \quad \dots, \quad X_{n-1} = tu^{n-1}, \quad X_n = u^n.$$

Also it is clear that a rational normal curve in \mathbf{P}^n is a nondegenerate curve of degree n , which implies that the points are in general position. Put $t := \lceil (\sum m_i + n - 2) / n \rceil$. If $t \leq m_1 + m_2 - 1$, the conclusion follows by Corollary 2 and Theorem 6. Hence we may assume that $t \geq m_1 + m_2$ and, as usual, that $\wp_s = (X_1, \dots, X_n)$. Further we let $J = \wp_1^{m_1} \cap \wp_2^{m_2} \cap \dots \cap \wp_{s-1}^{m_{s-1}}$. We claim that

$$X_0^{(t-1)-(m_s-1)} X_1^{m_s-1} \notin J + \wp_s^{m_s}.$$

In fact, if $X_0^{(t-1)-(m_s-1)} X_1^{m_s-1} \in J + \wp_s^{m_s}$, then for some $F \in [\wp_s^{m_s}]_{t-1} \subseteq \wp_s^{m_s-1}$ we have $H := X_0^{(t-1)-(m_s-1)} X_1^{m_s-1} + F \in J$. Since $X_0^{(t-1)-(m_s-1)} X_1^{m_s-1} \in \wp_s^{m_s-1}$, we get $H \in J \cap \wp_s^{m_s-1}$. By the definition of t and the remark before Theorem 6, we have $\sum m_i - 1 > n(t - 1)$, hence by Bezout's theorem we get that the hypersurface corresponding to H contains the rational curve C on which are our points. This implies that H must vanish on the point $(1, \alpha, \alpha^2, \dots, \alpha^n)$ for every $\alpha \in k$ and thus that $\alpha^{m_s-1} + F(1, \alpha, \alpha^2, \dots, \alpha^n) = 0$ for every $\alpha \in k$.

Since $F \in \wp_s^{m_s}$, we have $F(1, \alpha, \alpha^2, \dots, \alpha^n) = \alpha^{m_s} G(\alpha)$ for some polynomial $G \in k[X]$, hence we get $\alpha^{m_s-1} + \alpha^{m_s} G(\alpha) = 0$ for every $\alpha \in k$, a contradiction. This proves the claim. But then

$$X_0^{(t-1)-(m_s-1)} X_1^{m_s-1} \notin J + \wp_s^{m_s}.$$

This implies $r(R/(J + \wp_s^{m_s})) \geq t$. By Lemma 1 we get $r(R/I) \geq t$ and the conclusion follows from the above theorem.

Remark. The above bound for the regularity index can be attained also by fat points not lying on a rational normal curve (see [C1, §6]).

We end this section with the following result, which deals with the extremal case $r(R/I) = m_1 + m_2 - 1$ (see Corollary 2).

Corollary 8. *Let $n \geq 3$, $2 \leq s \leq n + 2$, and let P_1, \dots, P_s be distinct points in general position in \mathbf{P}_k^n . If $2 \leq m_1 \geq m_2 \geq \dots \geq m_s > 0$ are given integers and $I = \wp_1^{m_1} \cap \dots \cap \wp_s^{m_s}$, then $r(R/I) = m_1 + m_2 - 1$.*

Proof. By Corollary 2 and Theorem 6 we need only prove that $n(m_1 + m_2 - 1) \geq \sum m_i - 1$ or, equivalently,

$$(m_1 + m_2 - 1)(n - 1) - (m_3 + \dots + m_s) \geq 0.$$

But we have

$$\begin{aligned} & (m_1 + m_2 - 1)(n - 1) - (m_3 + \dots + m_s) \\ & \geq (m_1 + m_2 - 1)(n - 1) - nm_2 = (m_1 - 1)(n - 1) - m_2 \\ & \geq (m_1 - 1)(n - 1) - m_1 = (m_1 - 1)(n - 2) - 1. \end{aligned}$$

The conclusion follows from the assumptions $n \geq 3$ and $m_1 \geq 2$.

A GEOMETRIC INTERPRETATION

We recall that if Z is the zero-dimensional subscheme of \mathbf{P}_k^n corresponding to the ideal $I = \wp_1^{m_1} \cap \dots \cap \wp_s^{m_s}$, then for every integer t we can consider the linear system \mathcal{L}_t of all hypersurfaces of degree t containing Z as a subscheme. It is clear that

$$\begin{aligned} \dim_k(\mathcal{L}_t) &= \dim_k(I_t) - 1 = \dim_k(R_t) - H_{R/I}(t) - 1 \\ &= \dim_k(R_t) - e + h(\mathcal{L}_t) - 1, \end{aligned}$$

where $h(\mathcal{L}_t) := e - H_{R/I}(t)$ is called the *superabundance* of \mathcal{L}_t . The linear system \mathcal{L}_t is said to be *regular* iff $h(\mathcal{L}_t) = 0$, that is, $H_{R/I}(t) = e$ or, with the notation of the above sections, $r(R/I) \leq t$.

We fix the following notation: $J := \wp_1^{m_1} \cap \dots \cap \wp_s^{m_s}$, $I := J \cap \wp^a$, $I' := J \cap \wp^{a-1}$, $e = e(R/I)$, $e' = e(R/I')$. It is clear that

$$\begin{aligned} e &= \sum_{i=1}^s \binom{m_i + n - 1}{n} + \binom{a + n - 1}{n}, \\ e' &= \sum_{i=1}^s \binom{m_i + n - 1}{n} + \binom{a + n - 2}{n}, \end{aligned}$$

hence

$$e - e' = \binom{a + n - 2}{n - 1}.$$

By using the formula for the dimension of the sum of two vector spaces in terms of the dimensions of the summands and that of the intersection, we have

$$\begin{aligned} \dim_k(I'_t) - \dim_k(I_t) &= \dim(\wp^{a-1}/\wp^a)_t - \dim((J + \wp^{a-1})/(J + \wp^a))_t \\ &\leq \dim(\wp^{a-1}/\wp^a)_t \leq e - e'. \end{aligned}$$

Lemma 9. *The linear system \mathcal{L}_t is regular if and only if the linear system \mathcal{L}'_t is regular and $\dim_k(I'/I)_t \geq e - e'$.*

Proof. The conclusion follows immediately from the above remark and the exact sequence $0 \rightarrow I'/I \rightarrow R/I \rightarrow R/I' \rightarrow 0$.

Now it is clear that in order to prove Theorem 6 using induction on the sum of the multiplicities, we need only prove that there exist $e - e'$ forms in I'_t that are k -linearly independent modulo I_t . This can be proved as in Proposition 5.

This approach has a nice geometric interpretation. We shall see that the existence of such $e - e'$ forms is equivalent to the fact that the hypersurfaces of \mathcal{L}'_t separate the directions at the fat point P . To explain this fact we need some more notation.

Let π be the hyperplane $X_0 = 0$. We may consider π as \mathbf{P}_k^{n-1} with coordinate ring $S = k[X_1, \dots, X_n]$. Since

$$\dim_k(S_{a-1}) = \binom{n+a-2}{a-1} = e - e',$$

we may find $e - e'$ points in \mathbf{P}_k^{n-1} , say $Q_1, \dots, Q_{e-e'}$, that are not on a hypersurface of degree $a-1$ in \mathbf{P}_k^{n-1} . This implies that for every $i = 1, \dots, e - e'$, there is a form $G_i \in S_{a-1}$ such that $G_i(Q_i) \neq 0$ and $G_i(Q_j) = 0$ if $j \neq i$. We consider for every $i = 1, \dots, e - e'$ the line L_i connecting P and Q_i .

With this notation we say that the hypersurfaces of \mathcal{L}'_t separate the directions at the fat point P if there exist hypersurfaces $F_1, \dots, F_{e-e'}$ in \mathcal{L}'_t such that each F_i contains the subscheme aP of L_j for every $j \neq i$ and does not contain the subscheme aP of L_i .

Proposition 10. *There exist $e - e'$ elements in I'_t that are k -linearly independent modulo I_t if and only if the hypersurfaces of \mathcal{L}'_t separate the directions at the fat point P .*

Proof. Let $F_1, \dots, F_{e-e'}$ be the elements of I'_t that are k -linearly independent modulo I_t . Since $\dim_k(\wp^{a-1}/\wp^a)_t = e - e'$ and $F_1, \dots, F_{e-e'}$ are k -linearly independent also modulo \wp^a_t , they form a basis for the vector space $(\wp^{a-1}/\wp^a)_t$. The elements $G_1, \dots, G_{e-e'}$ defined above can be considered as elements of \wp^{a-1} . Hence for every $i = 1, \dots, e - e'$ we can write

$$X_0^{t-(a-1)}G_i = \sum \lambda_{ij}F_j + H_i$$

for suitable $\lambda_j \in k$ and $H_i \in \wp^a$. Due to the choice of $G_1, \dots, G_{e-e'}$ it is easy to see that for i running from 1 to $e - e'$, the hypersurfaces corresponding to $\sum \lambda_{ij}F_j$ separate the directions at the fat point P .

Conversely, let $F_1, \dots, F_{e-e'}$ be the elements of I'_t given by our assumption. If $\sum \lambda_i F_i \in I_t$, then $\sum \lambda_i F_i \in \wp^a$. It follows that $\sum \lambda_i F_i$ contains the subscheme aP of L_i for every i . This clearly implies $\lambda_i = 0$ for every i , and we are done.

REFERENCES

[A] J. Alexander, *Singularit es imposables en position g n ral   une hypersurface projective*, *Compositio Math.* **68** (1988), 305-354.
 [C] G. Castelnuovo, *Ricerche generali sopra i sistemi lineari di curve piane*, *Mem. Reale Accad. Sci. Torino* **42** (1891), 137-188.

- [C1] M. V. Catalisano, *Linear systems of lane curves through fixed "fat" points of \mathbf{P}^2* , J. Algebra **142** (1991), 81–100.
- [C2] —, *"Fat" points on a conic*, Comm. Algebra **19** (1991), 2153–2168.
- [CG] M. V. Catalisano and S. Greco, *Linear systems: developments of some results by F. Enriques and B. Segre*, Lecture Notes in Pure and Appl. Math., vol. 132, Dekker, New York, 1991, pp. 41–57.
- [DG] E. Davis and A. Geramita, *The Hilbert Function of a special class of 1-dimensional Cohen-Macaulay graded algebras*, The Curves Seminar at Queen's, Queen's Papers in Pure and Appl. Math. **67** (1984), 1H–29H.
- [G1] A. Gimigliano, *Regularity of linear systems of plane curves*, J. Algebra **124** (1989), 447–460.
- [G2] —, *Our thin knowledge on fat points*, The Curve Seminar at Queen's III, Queen's Papers in Pure and Appl. Math. **83** (1989), 1B–50B.
- [G] S. Greco, *Remarks on the postulation of zero-dimensional subschemes of projective space*, Math. Ann. **284** (1989), 343–351.
- [H] B. Harbourne, *Complete linear systems on rational surfaces*, Trans. Amer. Math. Soc. **289** (1985), 213–226.
- [Hi] A. Hirschowitz, *La méthode d'Horace pour l'Interpolation à plusieurs variables*, Manuscripta Math. **50** (1985), 337–388.
- [S] B. Segre, *Alcune questioni su insiemi finiti di punti in Geometria Algebrica*, Atti Convegno Internaz. Geometria Algebrica, Torino, 1961, pp. 15–33.

(N. V. Trung) INSTITUTE OF MATHEMATICS, BOX 631, BÓ HỒ, HANOI, VIETNAM

(M. V. Catalisano and G. Valla) DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI GENOVA, VIA
L. B. ALBERTI 4, 16132 GENOVA, ITALY
E-mail address, M. V. Catalisano: arezzo@igecuniv.bitnet
E-mail address, G. Valla: valla@igecuniv.bitnet