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INVARIANT MANIFOLDS OF HYPERCYCLIC VECTORS

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ABSTRACT. We show that any hypercyclic operator on Hilbert space has a dense, invariant linear manifold consisting, except for zero, entirely of hypercyclic vectors.

Recently, Beauzamy [1–3] constructed examples of linear operators on Hilbert space having dense, invariant linear manifolds all of whose nonzero elements are hypercyclic. Using different techniques, Godefroy and Shapiro [8] showed how to construct such manifolds consisting of cyclic, supercyclic, or hypercyclic vectors. In this note, we show that *any* hypercyclic operator on a Hilbert space must have a dense, invariant linear manifold consisting, except for zero, entirely of hypercyclic vectors. Interest in constructing linear manifolds of cyclic/hypercyclic vectors arises from the invariant subspace/subset problem for linear operators on Hilbert space. If *all* of the nonzero vectors in a Hilbert space were, say, hypercyclic for an operator T, then T would have no nontrivial, closed invariant subsets (and hence, no nontrivial invariant subspaces).

A vector f in the (complex) Hilbert space H is *hypercyclic* for the bounded linear operator $T: H \to H$ provided its orbit under T,

$$\mathrm{Orb}(T, f) := \{f, Tf, T^2 f, \dots\},$$

is dense in H. If the set of scalar multiples of the elements of $\mathrm{Orb}(T, f)$ is dense in H, then f is *supercyclic* for T; if the linear span of $\mathrm{Orb}(T, f)$ is dense in H, then f is *cyclic* for T. A bounded linear operator T on a Hilbert space is *hypercyclic, supercyclic*, or *cyclic* if it has, respectively, a hypercyclic, supercyclic, or cyclic vector. Hypercyclicity is a far more common phenomenon on Hilbert space than one might expect. For example, each of the following classes of linear maps contains hypercyclic operators: co-analytic Toeplitz operators [15, 8], compact perturbations of the identity [11, 6], translations [6], and composition operators [4, 5, 14]. Linear operators on finite-dimensional Hilbert space, however, are never hypercyclic, as the following proposition, due to Kitai [12], shows.

Proposition. *Suppose that T is hypercyclic on the Hilbert space H. Then the point spectrum of T^* is empty.*

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Proof. Suppose that T^* has an eigenvalue λ and that g is a corresponding eigenvector. Let f in H be arbitrary, and let $\langle \cdot , \cdot \rangle$ denote the inner product of H. Then

$$\{\langle g , T^n f\rangle : n = 0 , 1 , 2 , \dots\} = \{\lambda^n \langle g , f\rangle : n = 0 , 1 , 2 , \dots\}$$

is not dense in $\mathbf{C}$. It follows that f cannot be a hypercyclic vector; and because f was arbitrary, T cannot be hypercyclic. □

Of course, the proposition above may be restated: If T is hypercyclic, then $T-\lambda$ has dense range for any complex λ. That $T-\lambda$ has dense range whenever T is hypercyclic is the key to the proof of the following.

Theorem. *Suppose T is hypercyclic on the Hilbert space H. Then there is a dense invariant linear manifold of H consisting entirely, except for zero, of vectors that are hypercyclic for T.*

Proof. Let f be a hypercyclic vector for T. Following Godefroy and Shapiro [8], we note that

$$M = \{p(T)f : p \text{ is a polynomial}\}$$

is a dense linear manifold of H invariant under T (dense, because M contains the orbit of the hypercyclic vector f). We must show that every nonzero element of M is hypercyclic for T.

Let $p(T)f$ be an arbitrary element of M. As pointed out in [8], $p(T)f$ will be hypercyclic— $\text{Orb}(T , p(T)f)$ will be dense in H—provided $p(T)$ has dense range. To see this, observe that since T commutes with $p(T)$,

$$\text{Orb}(T , p(T)f) = p(T)\,\text{Orb}(T , f);$$

that is, the orbit of $p(T)f$ under T is the image of $\text{Orb}(T , f)$ under the map $p(T)$. Now, if $p(T)$ has dense range then $\text{Orb}(T , p(T)f)$ will be dense, being the image of the dense set $\text{Orb}(T , f)$ under an operator with dense range. Thus, to establish the theorem it suffices to show that $p(T)$ had dense range.

Express p as a product of linear factors; $p(T)$ is a scalar times a product of factors of the form $T - \lambda$. By the proposition, each of the factors $T - \lambda$ has dense range. Hence, $p(T)$ has dense range because it may be written as a product of operators with dense range. □

A supercyclic operator need not have an invariant manifold all of whose nonzero elements are supercyclic (or cyclic). Let T be hypercyclic on the Hilbert space H $(H \neq 0)$ and let f be a hypercyclic vector for T. As pointed out in [10], $T \oplus I$ is a supercyclic operator on $H \oplus \mathbf{C}$ with supercyclic vector $f \oplus 1$ (the closure of the orbit of $f \oplus 1$ under $T \oplus I$ is $H \oplus 1$ and it follows that the closure of the set of scalar multiples of the elements of $\text{Orb}(T \oplus I ,$ $f \oplus 1)$ is $H \oplus \mathbf{C}$). Suppose that $g \oplus \alpha$ is supercyclic or cyclic for $T \oplus I$. It easy to see that

$$g \oplus \alpha - (T \oplus I)(g \oplus \alpha) = (g - Tg) \oplus 0$$

is a nonzero vector that is not cyclic for $T \oplus I$. Hence, any nonzero invariant linear manifold for $T \oplus I$ contains nonzero vectors that are not cyclic.

Remarks. 1. The results presented above are valid in a Banach space setting (with essentially the same proofs).

2. The circle of ideas discussed in this paper yields a short, simple argument showing that the linear span of the collection of cyclic vectors of a cyclic operator is dense [13, Remark 4; 7; 9, problem 166]. Suppose T is a cyclic operator on a Hilbert (or Banach) space with cyclic vector f. Choose α large enough so that $T-\alpha$ is invertible (e.g., $\alpha > \|T\|$). Because T commutes with $(T-\alpha)^n$, $(T-\alpha)^n f$ is cyclic for $n = 0, 1, 2, \ldots$. Hence the linear span of $\mathrm{Orb}(T-\alpha, f)$ is contained in the linear span of the cyclic vectors for T; moreover, the linear span of $\mathrm{Orb}(T-\alpha, f)$, being equal to the linear span of $\mathrm{Orb}(T, f)$, is dense.

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