INVARIANT MANIFOLDS OF HYPERCYCLIC VECTORS

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Abstract. We show that any hypercyclic operator on Hilbert space has a dense, invariant linear manifold consisting, except for zero, entirely of hypercyclic vectors.

Recently, Beauzamy [1–3] constructed examples of linear operators on Hilbert space having dense, invariant linear manifolds all of whose nonzero elements are hypercyclic. Using different techniques, Godefroy and Shapiro [8] showed how to construct such manifolds consisting of cyclic, supercyclic, or hypercyclic vectors. In this note, we show that any hypercyclic operator on a Hilbert space must have a dense, invariant linear manifold consisting, except for zero, entirely of hypercyclic vectors. Interest in constructing linear manifolds of cyclic/hypercyclic vectors arises from the invariant subspace/subset problem for linear operators on Hilbert space. If all of the nonzero vectors in a Hilbert space were, say, hypercyclic for an operator $T$, then $T$ would have no nontrivial, closed invariant subsets (and hence, no nontrivial invariant subspaces).

A vector $f$ in the (complex) Hilbert space $H$ is hypercyclic for the bounded linear operator $T: H \rightarrow H$ provided its orbit under $T$,

$$\text{Orb}(T, f) := \{ f, Tf, T^2f, \ldots \},$$

is dense in $H$. If the set of scalar multiples of the elements of $\text{Orb}(T, f)$ is dense in $H$, then $f$ is supercyclic for $T$; if the linear span of $\text{Orb}(T, f)$ is dense in $H$, then $f$ is cyclic for $T$. A bounded linear operator $T$ on a Hilbert space is hypercyclic, supercyclic, or cyclic if it has, respectively, a hypercyclic, supercyclic, or cyclic vector. Hyercyclicitiy is a far more common phenomenon on Hilbert space than one might expect. For example, each of the following classes of linear maps contains hypercyclic operators: co-analytic Toeplitz operators [15, 8], compact perturbations of the identity [11, 6], translations [6], and composition operators [4, 5, 14]. Linear operators on finite-dimensional Hilbert space, however, are never hypercyclic, as the following proposition, due to Kitai [12], shows.

**Proposition.** Suppose that $T$ is hypercyclic on the Hilbert space $H$. Then the point spectrum of $T^*$ is empty.
Proof. Suppose that \( T^* \) has an eigenvalue \( \lambda \) and that \( g \) is a corresponding eigenvector. Let \( f \) in \( H \) be arbitrary, and let \( \langle \cdot, \cdot \rangle \) denote the inner product of \( H \). Then

\[
\{ \langle g, T^n f \rangle : n = 0, 1, 2, \ldots \} = \{ \lambda^n \langle g, f \rangle : n = 0, 1, 2, \ldots \}
\]

is not dense in \( C \). It follows that \( f \) cannot be a hypercyclic vector; and because \( f \) was arbitrary, \( T \) cannot be hypercyclic. □

Of course, the proposition above may be restated: If \( T \) is hypercyclic, then \( T - \lambda \) has dense range for any complex \( \lambda \). That \( T - \lambda \) has dense range whenever \( T \) is hypercyclic is the key to the proof of the following.

Theorem. Suppose \( T \) is hypercyclic on the Hilbert space \( H \). Then there is a dense invariant linear manifold of \( H \) consisting entirely, except for zero, of vectors that are hypercyclic for \( T \).

Proof. Let \( f \) be a hypercyclic vector for \( T \). Following Godefroy and Shapiro [8], we note that

\[
M = \{ p(T)f : p \text{ is a polynomial} \}
\]

is a dense linear manifold of \( H \) invariant under \( T \) (dense, because \( M \) contains the orbit of the hypercyclic vector \( f \)). We must show that every nonzero element of \( M \) is hypercyclic for \( T \).

Let \( p(T)f \) be an arbitrary element of \( M \). As pointed out in [8], \( p(T)f \) will be hypercyclic—\( \text{Orb}(T, p(T)f) \) will be dense in \( H \)—provided \( p(T) \) has dense range. To see this, observe that since \( T \) commutes with \( p(T) \),

\[
\text{Orb}(T, p(T)f) = p(T)\text{Orb}(T, f);
\]

that is, the orbit of \( p(T)f \) under \( T \) is the image of \( \text{Orb}(T, f) \) under the map \( p(T) \). Now, if \( p(T) \) has dense range then \( \text{Orb}(T, p(T)f) \) will be dense, being the image of the dense set \( \text{Orb}(T, f) \) under an operator with dense range. Thus, to establish the theorem it suffices to show that \( p(T) \) had dense range.

Express \( p \) as a product of linear factors; \( p(T) \) is a scalar times a product of factors of the form \( T - \lambda \). By the proposition, each of the factors \( T - \lambda \) has dense range. Hence, \( p(T) \) has dense range because it may be written as a product of operators with dense range. □

A supercyclic operator need not have an invariant manifold all of whose nonzero elements are supercyclic (or cyclic). Let \( T \) be hypercyclic on the Hilbert space \( H \ (H \neq 0) \) and let \( f \) be a hypercyclic vector for \( T \). As pointed out in [10], \( T \oplus I \) is a supercyclic operator on \( H \oplus C \) with supercyclic vector \( f \oplus 1 \) (the closure of the orbit of \( f \oplus 1 \) under \( T \oplus I \) is \( H \oplus 1 \) and it follows that the closure of the set of scalar multiples of the elements of \( \text{Orb}(T \oplus I, f \oplus 1) \) is \( H \oplus C \)). Suppose that \( g \oplus \alpha \) is supercyclic or cyclic for \( T \oplus I \). It easy to see that

\[
g \oplus \alpha - (T \oplus I)(g \oplus \alpha) = (g - Tg) \oplus 0
\]

is a nonzero vector that is not cyclic for \( T \oplus I \). Hence, any nonzero invariant linear manifold for \( T \oplus I \) contains nonzero vectors that are not cyclic.

Remarks. 1. The results presented above are valid in a Banach space setting (with essentially the same proofs).
2. The circle of ideas discussed in this paper yields a short, simple argument showing that the linear span of the collection of cyclic vectors of a cyclic operator is dense [13, Remark 4; 7; 9, problem 166]. Suppose $T$ is a cyclic operator on a Hilbert (or Banach) space with cyclic vector $f$. Choose $\alpha$ large enough so that $T - \alpha$ is invertible (e.g., $\alpha > \|T\|$). Because $T$ commutes with $(T - \alpha)^n$, $(T - \alpha)^n f$ is cyclic for $n = 0, 1, 2, \ldots$. Hence the linear span of $\text{Orb}(T - \alpha, f)$ is contained in the linear span of the cyclic vectors for $T$; moreover, the linear span of $\text{Orb}(T - \alpha, f)$, being equal to the linear span of $\text{Orb}(T, f)$, is dense.

References

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