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FOLIATIONS OF $E(5_2)$ AND RELATED KNOT COMPLEMENTS

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ABSTRACT. The depth one foliations with a single compact leaf are classified for twist knots and pretzel knots $(3, -3, 2n+1)$.

1. THE MAIN RESULTS

David Gabai's constructions [5, 6, 7, 8, 9] of taut foliations of knot complements $E(\kappa)$, when mastered, are simple and elegant but they do not give much insight into what the foliations look like. There are interesting classes of knots, however, for which a large class of depth one foliations can be explicitly classified and visualized. These include the twist knots, certain pretzel knots, and the arborescent knots defined by evenly weighted trees. This last class includes all two-bridge knots.

In this note, which is an introduction to the above program, we work with a couple of families of knots for which our constructions are particularly transparent. By avoiding a number of technical problems that arise in the more general case, we hope to make the central idea clearer.

The knots to be studied are the n-twist knots and the pretzel knots of type $(3, -3, 2n+1)$. These knots κ are of genus one and have a unique minimal genus Seifert surface S. When cut apart along S, $E(\kappa)$ becomes a manifold $X_S(\kappa)$ that is easily visualized, with taut, depth one foliations that are readily classified. The simplest of these knots, the 3-twist knot 5_2 (see the knot tables in Rolfsen [12]), can be taken as a "universal model" for the rest (Corollary 1.7). In [2] we will use surgery techniques to classify the depth one foliations of a broader class of knots of arbitrary genus.

For a knot $\kappa \subset S^3$, we let $N(\kappa)$ be a normal neighborhood of κ and denote the knot exterior by

$$E(\kappa) = S^3 \setminus \operatorname{int} N(\kappa).$$

Let $X_S(\kappa)$ be $E(\kappa)$ cut apart along a minimal genus Seifert surface S. Then

$$\partial X_S(\kappa) = S^- \cup A \cup S^+,$$

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where $S^{\pm}$ are two copies of S and the annulus A is formed by cutting $\partial N(\kappa)$ apart along $S \cap \partial N(\kappa)$. We take $I = [-1, 1]$ and let $M_S(\kappa)$ be $X_S(\kappa) \cup (D^2 \times I)$ with $\partial D^2 \times I$ pasted to the annulus A.

The manifold $E(\kappa)$ is determined by the knot κ and conversely (Gordon and Leucke [10]). By contrast, $M_S(\kappa)$ is not generally determined by κ (the choices of minimal genus Seifert surface may not all be isotopic) nor does it determine κ.

We are interested in smooth, taut, transversely oriented, finite depth foliations $\mathscr{F}$ of $E(\kappa)$, which meet $\partial E(\kappa)$ transversely. If $\mathscr{F}$ has all leaves compact, then these leaves are minimal genus Seifert surfaces and $E(\kappa)$ is fibered over the circle with the leaves as fibers. Otherwise we will assume that $\mathscr{F}$ is a depth one foliation with only one compact leaf S. It follows that S is a minimal genus Seifert surface (since $\mathscr{F}$ is taut [13]) and that every other leaf of $\mathscr{F}$ is proper, accumulating only on the leaf S and meeting $\partial E(\kappa)$ in circles [6].

The class of smooth-leaved, C^0 foliations is closed under the cut-and-paste constructions that we use. It is known that depth one foliations of this class are C^∞-smoothable [3], so we fix the assumption that all foliations are smooth-leaved and of class C^0.

The foliation $\mathscr{F}$ induces a depth one foliation $\mathscr{F}_X$ of $X_S(\kappa)$ with transverse boundary A and tangential boundary $S^- \cup S^+$. The corners $S^- \cap A$ and $S^+ \cap A$ of $\partial X_S(\kappa)$ are convex with respect to $\mathscr{F}_X$. The manifold $M_S(\kappa)$ has a natural foliation $\mathscr{F}_M$, assembled from $\mathscr{F}_X$ and the product foliation on $D^2 \times I$. The boundary of $M_S(\kappa)$ is $T^- \cup T^+$ where $T^- = S^- \cup (D^2 \times \{-1\})$ and $T^+ = S^+ \cup (D^2 \times \{1\})$. The foliation $\mathscr{F}_M$ is taut, transversely oriented, of depth one, and has $T^{\pm}$ as its only compact leaves. Finally, the arc $\gamma = \{0\} \times I \subset D^2 \times I$ is transverse to $\mathscr{F}_M$. For convenience, we will say that a foliation $\mathscr{F}_M$ with all of these properties is *admissible.*

Conversely, starting with an admissible foliation $\mathscr{F}_M$ of $M_S(\kappa)$, we excise an open normal neighborhood of γ to obtain a manifold homeomorphic to $X_S(\kappa)$, with boundary $S^- \cup A \cup S^+$ and corners $S^- \cap A$ and $S^+ \cap A$, together with a foliation $\mathscr{F}_X$ of depth one, tangent to $S^- \cup S^+$, and meeting A transversely in circles. A suitable gluing map $h\colon S^- \to S^+$ then yields $E(\kappa)$, together with a depth one foliation $\mathscr{F}$ that meets the boundary in circles and has one compact leaf S. These circles, which came from the circles on A, are longitudes of the knot. An arc across A becomes a meridian of the knot after the pasting. By an isotopy, one can choose the gluing map h so that this arc becomes a closed transversal to $\mathscr{F}$ meeting S. It is clear that the noncompact leaves also meet a closed transversal, so $\mathscr{F}$ is taut. In summary:

Lemma 1.1. *The correspondence $\mathscr{F} \to \mathscr{F}_M$ passes to a one-one correspondence between the set of homeomorphism classes of taut, transversely oriented, depth one foliations of $E(\kappa)$ having the Seifert surface S as unique compact leaf and the set of homeomorphism classes of admissible foliations $\mathscr{F}_M$ of $M_S(\kappa)$.*

The knots in this paper have minimal genus Seifert surface S unique up to isotopy, so we will drop the subscript "S" on $M(\kappa)$ and $X(\kappa)$. These knots also have genus one, so T^- and T^+ are tori. By the well-known classification of foliation germs at a toral leaf, the depth one hypothesis implies that, near $T^{\pm}$, the admissible foliation looks like the Reeb foliation. By removing a regular neighborhood of $T^{\pm}$, we obtain a foliation $\mathscr{F}'_M$ transverse to $T^{\pm}$ and of depth zero, hence a fibration of $M(\kappa)$ over the circle. Note that the arc $\gamma = \{0\} \times I \subset D^2 \times I$ is transverse to $\mathscr{F}'_M$.

Lemma 1.2. *The correspondence $\mathscr{F}_M \to \mathscr{F}'_M$ induces a one-one correspondence between the set of homeomorphism classes of admissible foliations of $M(\kappa)$ and the set of homeomorphism classes of transversely oriented fibrations of $M(\kappa)$ over S^1 with fibers transverse to $\partial M(\kappa) \cup \gamma$.*

Proof. The fibration $\mathscr{F}'_M$ is determined, up to homeomorphism, by the foliation $\mathscr{F}_M$. Indeed, it is elementary that the relative homology class $[F, \partial F] \in H_2(M(\kappa), \partial M(\kappa))$ of the fiber F is uniquely determined by $\mathscr{F}_M$ and, as Thurston remarks [13, p. 113], the fiber is incompressible and every incompressible surface representing $[F, \partial F]$ is isotopic to F. Thus, two fibrations $\mathscr{F}'_M$ and $\mathscr{F}''_M$, both arising from $\mathscr{F}_M$, can be assumed to have one fiber F in common. Cutting apart along that fiber produces a manifold homeomorphic in two different ways to $F \times I$, from which it follows that the two fibrations are homeomorphic.

Conversely, if one starts with a fibration $\mathscr{F}'_M$ of $M(\kappa)$, transverse to the boundary $T^+ \cup T^-$ and to γ, one can "spin" the foliation near $T^- \cup T^+$ to get an admissible foliation $\mathscr{F}_M$ of $M(\kappa)$. The direction of spin is determined canonically by the transverse orientations of $\mathscr{F}'_M$ and of $T^{\pm}$, so the foliation $\mathscr{F}_M$ is determined, up to homeomorphism, by $\mathscr{F}'_M$. □

Finally, note that the total space

$$M(\kappa) = F \times I/\{(x, 0) \equiv (f(x), 1)\}$$

of the fibration $\mathscr{F}'_M$ is determined by the isotopy class of the monodromy $f: F \to F$, where F is the typical fiber. In summary one has an elementary but basic result.

Theorem 1.3. *Let κ be a knot of genus one with S a minimal genus Seifert surface. Then every taut, transversely oriented, depth one foliation of $E(\kappa)$, with S as sole compact leaf, induces a fibration on $M(\kappa)$, transverse to $\partial M(\kappa) \cup \gamma$ and determined, up to homeomorphism, by the monodromy map $f: F \to F$. Conversely given a fibration on $M(\kappa)$, transverse to $\partial M(\kappa) \cup \gamma$ and with monodromy $f: F \to F$, one has a taut, transversely orientated foliation on $E(\kappa)$ of depth one having S as unique compact leaf. The foliation is determined, up to homeomorphism, by the monodromy $f: F \to F$ and the gluing map $h: S^- \to S^+$.*

Our basic strategy in foliating $E(\kappa)$ is to find $M(\kappa)$ and fibrations $\mathscr{F}'_M$. For higher genus knots the situation is more complicated (cf. [2]) because it is not possible to get a fibration of $M(\kappa)$ transverse to the boundary.

Let $P(1, k)$ denote a solid torus with an open wormhole drilled out, which goes around once meridianally and k times longitudinally. Section 2 gives a careful definition of $P(1, k)$ and Figure 1 gives a picture of $P(1, 2)$ (see the next page).

Theorem 1.4. *For $k = 1, 2, 3, \ldots,$ the n-twist knots κ with $n = 2k - 1$ or $2k$ half-twists have $M(\kappa)$ homeomorphic to $P(1, k)$.*

See [1, p. 136] for a definition of the n-twist knot.

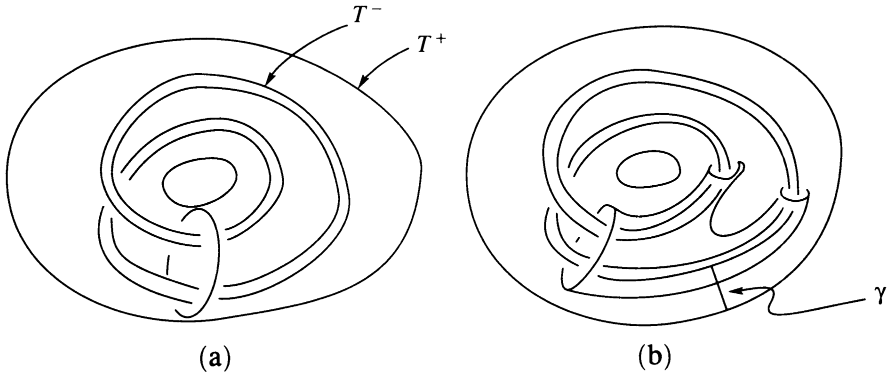

FIGURE 1. (a) $P(1, 2)$ fibered (b) $P(1, 2)$ foliated

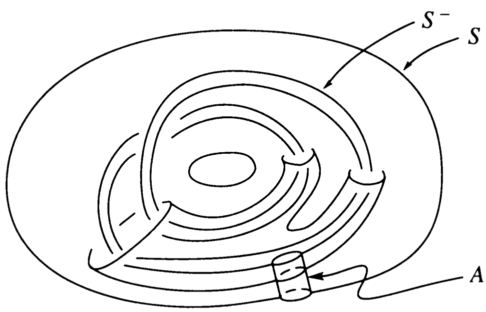

FIGURE 2. $X(5_2) = X(6_1)$

As an example of Theorem 1.4, consider the knots 5_2 and 6_1 [12, p. 391], which have 3 and 4 half twists respectively. The manifold $M(5_2) = M(6_1) = P(1, 2)$ is fibered with the indicated pair of pants surface (the planar surface with three boundary circles) as fiber (Figure 1(a)). The fibration, spun at the two boundary tori as above, becomes a depth one foliation of $P(1, 2)$ tangent to the boundary (Figure 1(b)). This foliation is just the Reeb foliation of the solid torus spun along the boundary of the wormhole and is transverse to γ. Excising an open normal neighborhood of γ produces a space homeomorphic to $X(5_2) = X(6_1)$ (Figure 2). Suitable gluings of the inner punctured torus to the outer one give foliations on $E(5_2)$ and $E(6_1)$. In particular, $E(5_2)$ is identical to $E(6_1)$ except for this last gluing map.

Theorem 1.5. *For $n = 0, 1, 2, \ldots$, the pretzel knot $\kappa = (3, -3, 2n+1)$ has $M(\kappa)$ homeomorphic to $P(1, 2)$.*

In particular, 9_{46} [12, p. 399], which is the pretzel knot $(3, -3, 3)$, has $M(9_{46}) = P(1, 2)$. See [7, p. 119] for a definition of pretzel knots. Thus, one has countably many knots κ with $M(\kappa) = P(1, 2)$. These knots all have F the pair of pants surface and monodromy f a rotation of F through π radians.

Their exteriors $E(3, -3, 2n+1)$ differ only in the gluing map $h: S^- \to S^+$, much as two fibered knots of a given genus have exteriors differing only in their monodromy.

The method used in the proof of Theorem 1.5 gives many more examples of knots κ with $M(\kappa) = P(1, k)$. For a fixed k, the complements of these knots differ only in the final gluing map of S^- to S^+. For all of these knots, one has a simple foliation with one compact leaf, which is basically the Reeb foliation on the solid torus.

But $P(1, k)$ has countably many other fibrations, so all these knot complements have countably many taut foliations of depth one. One classifies these fibrations by a theorem of Thurston [13, Theorem 5]. In the current situation, the fibrations could also be obtained directly by elementary means (see the remark on p. 9), but Thurston's theorem would still be needed to show that these constructions give all fibrations.

Theorem 1.6. *For $k = 2, 3 \ldots$ and every pair p and q of relatively prime integers with $q \neq 0$, there is a natural fibration of $P(1, k)$, transverse to $\partial P(1, k)$. The fiber is homeomorphic to a surface with $\gcd(p, k)$ boundary components lying on T^-, $\gcd(p+q, k)$ boundary components lying on T^+, and genus*

$$g = (2 + |q|(k-1) - \gcd(p, k) - \gcd(p+q, k))/2.$$

Finally, all fibrations of $P(1, k)$, transverse to the boundary, are so obtained, up to homeomorphism, and can be taken to be transverse to the arc γ (Figure 1(b)).

Remark. By the previous three theorems, Theorem 1.6 classifies, up to homeomorphism, the depth one foliations of $E(\kappa)$ that have the minimal genus Seifert surface as unique compact leaf, where κ is an arbitrary twist knot or a pretzel knot $(3, -3, 2n+1)$.

By Theorem 1.6,

$$P(1, k) = (F \times I)/\{(x, 0) \equiv (f(x), 1)\}$$

for countably many different choices of fiber F and corresponding monodromy f. If κ is the n-twist knot with $n = 2k-1$ or $2k$ half twists, then $M(\kappa) = P(1, k)$. In Theorem 1.6, if p and q are chosen so that $(p, k) = 1 = (p + q, k)$, then F will have one end on T_-, the other end on T_+, and F will have genus g as specified in Theorem 1.6. For the knot 5_2, if p is chosen odd and $q = 2g$, one obtains the same F. Thus the exteriors $E(\kappa)$ and $E(5_2)$ differ only in the monodromy $f: F \to F$ and gluing map $h: S^- \to S^+$. By spinning along the boundary, one obtains a taut, depth one foliation $\mathscr{F}$. The leaves of $\mathscr{F}$ in $U = E(\kappa)\backslash S$ are homeomorphic to $\operatorname{int}(F)$ and fiber U over S^1 with monodromy an end-periodic map [4], again denoted by $f: L \to L$. The manifold $E(\kappa)$ can be reassembled, up to homeomorphism, by giving the maps $h: S \to S$ and $f: L \to L$ up to isotopy.

Corollary 1.7. *Given a particular n-twist knot κ or pretzel knot*

$$\kappa = (3, -3, 2n+1),$$

there exists an open surface L with two ends and genus depending on κ so that one choice of the maps $f: L \to L$ and $h: S^- \to S^+$ (as above) yields $E(\kappa)$ and another choice of these gluing maps yields $E(5_2)$.

Thus the knot 5_2 is, in a certain sense, universal among these knots. With a little more work one also gets the same result for the genus one 2-bridge knots [1, p. 23].

2. Proofs of the theorems

Let D be the unit disc and consider b smaller open discs with centers evenly spaced around the circle of radius $3/4$. Deleting these open discs produces a planar surface D_b with $b+1$ boundary components. For example, D_2 is the pair of pants surface. Let $f\colon D_b \to D_b$ be rotation through $2\pi a/b$ radians, where a and b are relatively prime integers, and set

$$P = P(a, b) = (D_b \times I)/\{(x, 0) \equiv (f(x), 1)\}.$$

Then P is a solid torus with a wormhole drilled out that goes around a times meridianally and b times longitudinally. The manifold P has a fibration $\mathscr{F}_P'$, with fiber $F = D_b$ meeting the outer boundary component T^+ in a single circle C_0 and the inner boundary component T^- in circles $C_1, \dots, C_b$. If one spins $\mathscr{F}_P'$ along ∂P so as to be tangent to the boundary and transverse to a radial arc γ from T^- to T^+, one obtains a depth one foliation $\mathscr{F}_P$ of P (see Figure 1(b)). This foliation is just the Reeb foliation on the solid torus spun along the inner wormhole.

Let J_i be the radial arc from the ith inner boundary circle C_i to the outer boundary C_0. Then the image B of $\bigcup_{i=1}^b J_i \times I$ in $P = P(a, b)$, under the identification $(x, 0) \equiv (f(x), 1)$, is an annulus meeting the fibration $\mathscr{F}_P'$ in arcs parallel to the J_i and meeting $\mathscr{F}_P$ in a depth one foliation with the boundary circles as compact leaves. Cutting $P(a, b)$ apart along B yields a solid torus $T(a, b)$ with the two copies, B^- and B^+, of the annulus B, both running a times meridianally and b times longitudinally around the boundary of $T(a, b)$. The space $T(a, b)$ is, in fact, a sutured manifold in the sense of Gabai [6] with the two annuli, B^- and B^+, as sutures. The foliation of the sutured manifold $T(a, b)$, induced by $\mathscr{F}_P$, is a generalized version of the foliation constructed by Gabai in [6, Example 5.1].

We will need a description of the homology $H_2(P, \partial P)$. This manifold is the complement of a 2-component link $\lambda_1 \cup \lambda_2 \subset S^3$, so an elementary application of excision and the exact sequence of a pair gives

$$H_2(P, \partial P) = H_2(S^3, \lambda_1 \cup \lambda_2) = H_1(\lambda_1 \cup \lambda_2) = \mathbb{Z}^2.$$

Lemma 2.1. *The classes* $[B, \partial B]$, $[F, \partial F] \in H_2(P, \partial P)$ *constitute a free basis.*

Proof. Let $\sigma \subset T^-$ be a meridian (relative to the wormhole) and let $\tau \subset T^+$ be a longitude (relative to the solid torus before removing the wormhole). Then, with appropriate choice of orientations, the intersection product

$$H_1(P) \times H_2(P, \partial P) \to H_0(P) = \mathbb{Z}$$

gives

$$\begin{aligned} [\sigma]\cdot[F] &= 0 \qquad & [\sigma]\cdot[B] &= 1 \\ [\tau]\cdot[F] &= 1 \qquad & [\tau]\cdot[B] &= a. \quad \square \end{aligned}$$

Proof of Theorem 1.4. The n-twist knots κ are disc decomposable in the sense of Gabai [6]. The general cases in which $n = 2k-1$ or $2k$ half-twists are

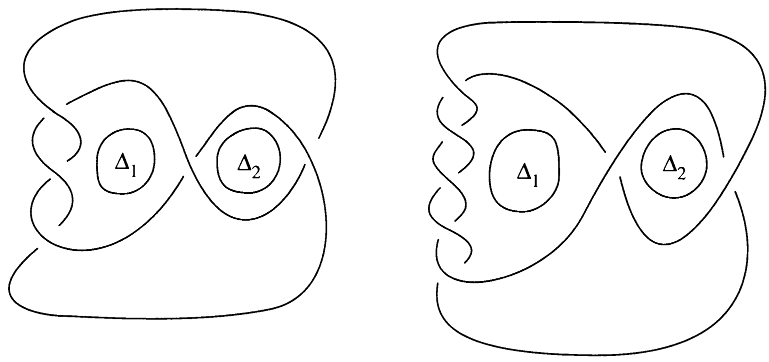

FIGURE 3. 5_2 and 6_1

well represented by the special cases $\kappa = 5_2$ $(n = 3)$ and $\kappa = 6_1$ $(n = 4)$. These are illustrated in Figure 3, where we indicate the minimal genus Seifert surface S and the discs Δ_1 and Δ_2 to be used in the disc decompositions. As a sutured manifold, $X(\kappa)$ is a 2-holed solid torus with one annular suture. The disc Δ_2 meets the suture twice. The decomposition of $X(\kappa)$ along Δ_2 yields a solid torus with a pair of parallel sutures running around once meridianally and k times longitudinally (in Figure 3, $k = 2$). That is, decomposing along Δ_2 produces the sutured manifold $T(1, k)$. Each of the sutures $B^{\pm}$ should be viewed as the union of two rectangles, one of which is the appropriate copy $\Delta_2^{\pm}$ of the decomposing disc.

In $P(1, k)$ (Figure 1), suppose that γ is the radial arc J_1 from T^- to T^+. Let $X(1, k)$ be $P(1, k)$ with a small tubular neighborhood of γ deleted. Let $D = B \cap X(1, k)$. Then, cutting $X(1, k)$ apart along the disc D yields the sutured manifold $T(1, k)$. Again, each suture is the union of two rectangles, one of which is the appropriate copy $D^{\pm}$ of the disc.

Thus, for κ the n-twist knot with $2k - 1$ or $2k$ half twists, $X(\kappa)$ cut apart along Δ_2 is homeomorphic, as a sutured manifold, to $X(1, k)$ cut apart along D, and the homeomorphism can be chosen so as to match $\Delta_2^{\pm}$ to $D^{\pm}$. It follows that $X(\kappa)$ is $X(1, k)$ and that $M(\kappa)$ is $P(1, k)$. □

Proof of Theorem 1.5. Figure 4 on the next page depicts the knot $6_1 = (3, -3, 1)$ and its Seifert surface S. The curve C on S (Figure 4) is homotopic, in the complement of S, to the curve C'. By Theorem 1.4, $E(6_1)$ is obtained from $X(1, 2)$ by gluing S^- to S^+. Since C has an untwisted annular neighborhood on S, $1/n$ surgery on C (see [12, pp. 258–260]) only changes the gluing map of S^- to S^+ (see Harer [11]). On the other hand, since C is unknotted, the surgery converts 6_1 to a knot in S^3 and, since C is homotopic to C', that knot is $(3, -3, 2n + 1)$. □

Proof of Theorem 1.6. The group $H_2(P, \partial P)$ sits as the integer lattice in $H_2(P, \partial P; \mathbb{R}) = \mathbb{R}^2$. We coordinatize this lattice by

$$[B, \partial B] = (1, 0) \in \mathbb{Z}^2, \qquad [F, \partial F] = (0, 1) \in \mathbb{Z}^2.$$

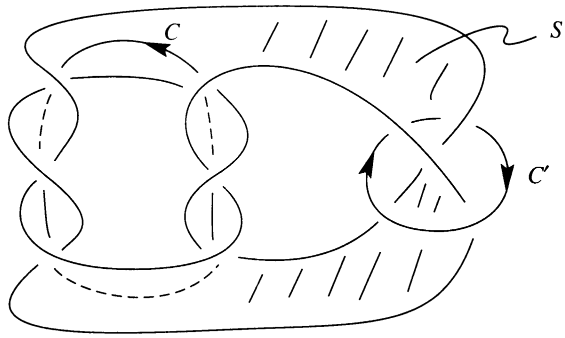

FIGURE 4. 6_1

The Thurston "norm" is a seminorm x on $H_2(P, \partial P; \mathbb{R})$ [13] and

$$x(1, 0) = 0,$$
$$x(0, 1) = k - 1,$$

these being $|\chi(B)|$ and $|\chi(F)|$, respectively. Therefore, the "unit ball" of x has boundary the pair of horizontal lines $y = \pm 1/(k-1)$. By [13, Theorem 5], a properly imbedded, connected, incompressible surface $\Sigma \subset P$ can only be the fiber of a fibration $P \to S^1$ if the ray out of $0 \in H_2(P, \partial P; \mathbb{R})$ and passing through $[\Sigma, \partial\Sigma]$ meets the interior of a face of the unit ball of x, in which case all (nondivisible) lattice points on rays through the interior of that face are represented by the (connected) fibers of fibrations. Here the nondivisibility and connectedness are equivalent and mean that the coordinates of the lattice points are relatively prime. By the proof of Lemma 1.2, the fibration corresponding to such a lattice point is unique up to homeomorphism.

Since $(0, 1)$ corresponds to the fiber F of a fibration, it follows that all lattice points (p, q) with relatively prime coordinates and $q > 0$ correspond to the connected fiber F' of a fibration. The ones with $q < 0$ correspond to the same fibrations with opposite transverse orientation. Thus, the only relatively prime pairs (p, q) that do not correspond to a fibration are $(\pm 1, 0)$ (the annulus B is not a fiber, as intuition suggests). If $[F', \partial F'] = (p, q)$, it remains for us to compute the number of components of $F' \cap T^-$, of $F' \cap T^+$, and the genus g of F'.

Let $\pi^\pm: H_1(\partial P) \to H_1(T^\pm)$ be a projection onto the appropriate summand and define

$$\partial^\pm = \pi^\pm \circ \partial_*: H_2(P, \partial P) \to H_1(T^\pm),$$

where ∂_* is the connecting homomorphism in the exact sequence of the pair $(P, \partial P)$.

Points of the lattice $H_1(T^-)$ are given coordinates $(r, s)^-$ as follows. It is clear that $\partial^-[B, \partial B]$ and the class $[m^-]$ of the meridian on T^- (relative to the wormhole) form a basis of $H_1(T^-)$. Set $[m^-] = (0, 1)^-$ and $\partial^-[B, \partial B] = (1, 0)^-$. Then $\partial^-[F, \partial F] = (0, k)^-$ and

$$\partial^-[F', \partial F'] = \partial^-(p[B, \partial B] + q[F, \partial F]) = (p, qk)^-,$$

implying that the number of boundary circles of the surface F' lying on T^- is $\gcd(p, qk)$. This is equal to $\gcd(p, k)$ since $\gcd(p, q) = 1$.

Coordinatize $H_1(T^+)$ by a meridian, denoted $(0, 1)^+$, and a longitude, denoted $(1, 0)^+$. Thus, from the picture of $P(1, k)$, $\partial^+[B, \partial B] = (k, 1)^+$ and $\partial^+[F, \partial F] = (0, 1)^+$. Therefore

$$\partial^+[F', \partial F'] = \partial^+(p[B, \partial B] + q[F, \partial F]) = (pk, p+q)^+$$

and the number of boundary circles of the surface F' lying on T^+ is $\gcd(pk, p+q)$. This is equal to $\gcd(k, p+q)$ since $\gcd(p, q) = 1$.

Since B is an annulus, the Thurston norm is

$$x([F', \partial F']) = x(p[B, \partial B] + q[F, \partial F]) = |q|x([F, \partial F]) = |q|(k-1),$$

and it follows that the Euler characteristic $\chi(F') = |q|(1-k)$. But $\chi(F') = 2 - 2g - b$ where g is the genus of F' and b is the number of boundary components. Thus $|q|(1-k) = 2 - 2g - \gcd(p, k) - \gcd(p+q, k)$ from which the formula for g given in the theorem follows.

Finally, the transversality of the arc γ (Figure 1(b)) can be seen in a number of ways. One way uses the remark immediately following this proof. Another uses differential forms. The fibration with fiber F is clearly transverse to (a suitably isotoped) γ, so a closed, nowhere vanishing 1-form ω, defining this fibration, does not vanish on $\dot{\gamma}$. Let η be a closed 1-form, Poincaré dual to B, supported in a regular neighborhood W of B, and transverse in W to ω. One can assume that γ misses W. Then $p\eta + q\omega$ is closed, nonsingular, and its de Rham class is Poincaré dual to $[F', \partial F']$. This form defines the fibration corresponding to (p, q) and does not vanish on $\dot{\gamma}$. $\square$

Remark. The fibration corresponding to the lattice point (p, q) admits an intuitive description. Cutting $P(1, k)$ open along B produces $T(1, k)$, fibered over S^1 with fiber a $4k$-gon G having edges $e_1, \dots, e_{4k}$. Write $P(1, k) = T(1, k)/\sim$, where $\sim$ pastes B^- to B^+. Under this pasting $F = D_k = G/\sim$ where $\sim$ identifies e_{4i-3} to e_{4i-1}, $1 \le i \le k$. The edges of the form e_{4i-2} become circles $C_1, \dots, C_k$ on T^- and the union of the edges of the form e_{4i} form a circle C_0 on T^+. Thus, G has k edges whose images under $\sim$ lie on T^- and k edges whose images lie on T^+. If, however, one reglues via a rotation of B through $2\pi p/qk$ radians, the resulting 3-manifold is again $P(1, k)$, but q distinct $4k$-gons are now assembled into the fiber F' of the new fibration. The reader is invited to think through the combinatorics that count the number of boundary components of F' on each of the tori $T^{\pm}$ and to compute the genus.

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