PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 118, Number 3, July 1993

A COUNTEREXAMPLE TO THE INFINITY VERSION OF THE HYERS AND ULAM STABILITY THEOREM

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(Communicated by Andrew M. Bruckner)

ABSTRACT. Hyers and Ulam proved a stability result for convex functions, defined in a subset of $\mathfrak{R}^n$. Here we give an example showing that their result cannot be extended to those functions defined in infinite-dimensional normed spaces. Also, we give a positive result for a particular class of approximately convex functions, defined in a Banach space, whose norm satisfies the so-called convex approximation property.

1. INTRODUCTION

In this paper we discuss the following problem: let Δ be a convex subset of a Banach space X. Consider an arbitrary ε-convex function $f: \Delta \to \mathfrak{R}$, that is, a function that satisfies for every x, $y \in \Delta$ and every $\lambda \in [0, 1]$ the inequality:

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) + \varepsilon.$$

Is it true that there exists a convex function $g: \Delta \to \mathfrak{R}$ such that $|g(x) - f(x)| \leq K\varepsilon$ $\forall x \in \Delta$, where K is a constant depending only on X? A positive answer was given by Hyers and Ulam [HU] (see also [Gr, C]) in the case $X = \mathfrak{R}^n$ (with any norm!); the best known estimates are with $K_n = \min(M_n, L_n)$ where $M_n = (n^2+3n)/(4n+4)$ and $L_n = m/2$ for $2^{m-1} \leq n < 2^m$ (see also [Ge] for a discussion concerning these constants and related questions). We will show that in the infinite-dimensional case the stability theorem of [HU] does not hold. Also we shall give a positive answer when the space X and the functions satisfy some additional properties. For midpoint convex functions a counterexample is known [Ge]; for a positive result concerning η-approximately convex functions, which are also ε-subadditive, see [K].

2. COUNTEREXAMPLE

Let X be a Banach space. We denote by $B(X)$ (resp. $B_r(X)$) the unit ball (resp. ball of radius r) of X, i.e., $B(X) = \{x \in X: \|x\| \leq 1\}$ (resp.

Received by the editors November 20, 1991.
1991 *Mathematics Subject Classification*. Primary 26B25; Secondary 46B99, 52A27.
Key words and phrases. Approximately convex functions, Hyers and Ulam theorem, convex approximation property.
During the preparation of the paper the authors were supported by grants from GNAFA-CNR and National Research Group "Functional Analysis" (M.U.R.S.T.).

$B_r(X) = \{x \in X : \|x\| \le r\})$, and if A is a subset of X, by $\operatorname{co} A$ its convex hull.

We say that the set $A \subseteq X$ satisfies the condition $\mathbf{C}^{\mathbf{o}}(\varepsilon)$ if x, $y \in A$ implies $(x+y)/2 \in A + B_\varepsilon(X)$. We say that the set A satisfies the condition $\mathbf{c}^{\mathbf{o}}(\varepsilon)$ if x, $y \in A$ and $\lambda \in [0, 1]$ implies $\lambda x + (1-\lambda)y \in A + B_\varepsilon(X)$.

Let l_1 be the Banach space of absolutely convergent sequences with the usual norm (that we will indicate by $\|x\|_1$). We will denote by $\{e_i\}$ the standard basis of l_1. Let $C^+ = \{x \in l_1 : x_i \ge 0\}$, the positive cone of l_1, and $B^+ = C^+ \cap B(l_1)$. For $p \in (0, 1)$ set $S_p = \{x \in B^+ : \sum_{i=1}^{+\infty} x_i^p \le 1\}$. We need the following two propositions: the first one is in [L], but we give the proof since the paper is not very accessible, the second is in [CP].

Proposition 1. *The set S_p satisfies condition $\mathbf{C}^{\mathbf{o}}(2^{1/p-1} - 1)$.*

Proof. Let x, $y \in S_p$ such that $\sum_{i=1}^{+\infty} x_i^p = \sum_{i=1}^{+\infty} y_i^p = 1$. Then we have

$$\begin{aligned} \inf_{z \in S_p} \left\| \frac{x+y}{2} - z \right\|_1 &= \inf_{\sum_{i=1}^{+\infty} z_i^p = 1} \sum_{i=1}^{+\infty} \left| \frac{x_i + y_i}{2} - z_i \right| \\ &= \inf_{\sum_{i=1}^{+\infty} z_i^p = 1 \ 2z_i \le x_i + y_i} \sum_{i=1}^{+\infty} \left(\frac{x_i + y_i}{2} - z_i \right). \end{aligned} \tag{2.1}$$

For positive numbers, $s^p + t^p = z^p$ implies $s + t \le z$; thus

$$\inf_{\sum_{i=1}^{+\infty} z_i^p = 1} \sum_{i=1}^{+\infty} \left(\frac{x_i + y_i}{2} - z_i \right) \le \inf_{\sum_{i=1}^{+\infty} s_i^p + t_i^p = 1} \sum_{i=1}^{+\infty} \left(\frac{x_i}{2} - s_i \right) + \left(\frac{y_i}{2} - t_i \right). \tag{2.2}$$

Now let x and y be in S_p; define x', y' (still in S_p and with disjoint supports) as follows:

$$x'_{2n} = 0, \qquad x'_{2n-1} = x_n, \qquad y'_{2n-1} = 0, \qquad y'_{2n} = y_n.$$

Then, by using (2.1) and (2.2), we obtain

$$\inf_{z \in S_p} \left\| \frac{x+y}{2} - z \right\|_1 \le \inf_{z \in S_p} \left\| \frac{x'+y'}{2} - z \right\|_1,$$

so we can suppose that x and y have disjoint supports. With this restriction, letting $z_i = (x_i + y_i)/2^{1/p}$ we obtain

$$\begin{aligned} \inf_{z \in S_p} \left\| \frac{x+y}{2} - z \right\|_1 &\le \left(\frac{1}{2} - \frac{1}{2^{1/p}} \right) \sum_{i=1}^{+\infty} (x_i + y_i) \\ &\le \left(\frac{1}{2} - \frac{1}{2^{1/p}} \right) \left(\sum_{i=1}^{+\infty} (x_i + y_i)^p \right)^{1/p} = 2^{1/p-1} - 1. \end{aligned}$$

Since x and y are arbitrary, S_p satisfies condition $\mathbf{C}^{\mathbf{o}}(2^{1/p-1} - 1)$.

Proposition 2 (see [CP]). *If A satisfies $\mathbf{C}^{\mathbf{o}}(\varepsilon)$ then A satisfies $\mathbf{c}^{\mathbf{o}}(2\varepsilon)$.*

So, in particular, S_p satisfies condition $\mathbf{c}^{\mathbf{o}}(2^{1/p} - 2)$.

Now let us give the counterexample. For each $p \in (0, 1)$, let $f_p \colon B^+ \subset l^1 \to \mathfrak{R}$ be defined in the following way: $f_p(x) = \operatorname{dist}(x, S_p) = \inf_{z \in S_p} \|x - z\|_1$. First of all we will prove that f_p is a $(2^{1/p} - 2)$-convex function. Take $\eta > 0$,

x, $y \in B^+$, and z_1, $z_2 \in S_p$ such that $\|x - z_1\|_1 \le f_p(x) + \eta$ and $\|y - z_2\|_1 \le f_p(y) + \eta$; then, by using Proposition 2, we obtain

$$\begin{aligned} f_p(\lambda x + (1-\lambda)y) &= \inf_{z \in S_p} \|\lambda x + (1-\lambda)y - z\|_1 \\ &\le \inf_{z \in S_p} (\lambda \|x - z_1\|_1 + (1-\lambda)\|y - z_2\|_1 \\ &\qquad + \|\lambda z_1 + (1-\lambda)z_2 - z\|_1) \\ &\le \lambda f_p(x) + (1-\lambda) f_p(y) + \eta + \inf_{z \in S_p} \|\lambda z_1 + (1-\lambda)z_2 - z\|_1 \\ &\le \lambda f_p(x) + (1-\lambda) f_p(y) + \eta + 2^{1/p} - 2. \end{aligned}$$

The conclusion follows from the arbitrariness of η.

Now suppose that there exists K such that there exists a convex function $g\colon B^+ \to \mathfrak{R}$ satisfying $|f_p(x) - g(x)| \le K(2^{1/p} - 2)$. Take $\overline{p} \in (0, 1)$ such that $K(2^{1/\overline{p}} - 2) \le 1/4$. Let

$$x_n = (\underbrace{1/n, \dots, 1/n}_{n}, 0, 0, \dots);$$

we have (since $f_p(e_i) = 0$)

$$f_{\overline{p}}(x_n) \le \frac{1}{4} + g(x_n) \le \frac{1}{4} + \sum_{i=1}^{n} \frac{g(e_i)}{n} \le \frac{1}{4} + \sum_{i=1}^{n} \frac{f_{\overline{p}}(e_i) + 1/4}{n} = \frac{1}{2}.$$

This is a contradiction, since an easy calculation shows that $f_{\overline{p}}(x_n) = 1 - n^{1-1/\overline{p}} \to 1$ as $n \to \infty$.

Remark 1. The finite-dimensional version of this counterexample shows that, at least asymptotically, the constants K_n that appear in [C] are best possible. In fact, define H_n as the best constant such that for every convex subset Δ of $\mathfrak{R}^n$ and for every ε-convex function f on Δ there exists a convex function g on Δ such that $|g(x) - f(x)| \le H_n \varepsilon$ for every $x \in \Delta$. Consider $S_p^n = \{x \in \mathfrak{R}^n : x_i \ge 0,\ \sum_{i=1}^n x_i^p \le 1\}$, and define, as before, $f_p(x) = \inf_{z \in S_p^n} \|x - z\|_1$. Then, if $x_n = (1/n, \dots, 1/n)$, proceeding in the same way as before, we obtain

$$\begin{aligned} 1 - n^{1-1/p} &= f_p(x_n) \le H_n(2^{1/p} - 2) + g(x_n) \\ &\le H_n(2^{1/p} - 2) + \sum_{i=1}^{n} \frac{g(e_i)}{n} \\ &\le H_n(2^{1/p} - 2) + \sum_{i=1}^{n} \frac{f_p(e_i) + H_n(2^{1/p} - 2)}{n} \\ &= 2H_n(2^{1/p} - 2). \end{aligned}$$

This implies

$$H_n \ge \frac{1 - n^{1-1/p}}{4(2^{1/p-1} - 1)} \sim \frac{\log_2 n}{4} \quad \text{as } p \to 1^-.$$

Remark 2. Now consider the Hilbert space $l_2 = \{\{x_n\} : \sum_{n=1}^{+\infty} x_n^2 < +\infty\}$ with the usual norm. Then the previous example works, also, if one thinks of B^+ as a subset of l_2. Notice that the convex set B^+ is a convex subset of l_2 (since algebraically $l_1 \subset l_2$) and, also, it is a closed subset in l_2. Take, in fact, a

sequence $\{x_n\} \subset B^+$ such that $x_n \to x$ in the l_2 norm. Then $\{x_n\}$ converges to x in the weak-topology of l_2 and so it converges coordinatewise. But, also, as a bounded sequence in l_1 it has a subnet converging in the weak*-topology of l_1 (and in particular coordinatewise) to an element of l_1. So x belongs to l_1, thus to B^+.

Remark 3. As the previous remark pointed out, the stability problem is (in some sense) independent of the norm-topology of the Banach space in which the domain of our ε-convex functions lie. A way to relate the norm to the functions is to ask that they must satisfy some lipschitz condition. Notice that the functions f_p considered in our counterexample are 1-lip, since it follows easily, from the definition of distance, that $|f_p(x) - f_p(y)| \leq \|x - y\|_1$. The same condition is not satisfied if we think of our example embedded in l_2. In fact, as a simple calculation shows, for any fixed h our functions are not h-lip, for every p, in the l_2-norm. This will follow directly from Theorem 1 of the next section.

3. A POSITIVE RESULT

In this section, we prove that the construction of our counterexample is possible since l_1 lacks a convexity property called the "convex approximation property" (C.A.P. for short). We will say that a Banach space X has C.A.P. if for every $\varepsilon > 0$ and $r > 0$, there exists an integer $p = p(\varepsilon, r)$ such that for every $A \subseteq B_r(X)$ we have $\operatorname{co} A \subseteq \operatorname{co}_p A + B_\varepsilon(X)$, where $\operatorname{co}_p A = \{x \in X : x \in \sum_{i=0}^p \alpha_i x_i,\ x_i \in X,\ \alpha_i \geq 0,\ \sum_{i=0}^p \alpha_i = 1\}$. In other words, each element of $\operatorname{co} A$ can be ε-approximated by a convex combination of no more than p vectors of A. We will say that X is B-convex [P] if there exist constants $c > 0$, $p > 1$ such that for every n and all independent random variables $g_1, \dots, g_n$ with values in X we have

$$E\left(\left\|\sum_{i=1}^n g_i\right\|_X^p\right) \leq c^p \sum_{i=1}^n E\|g_i\|_X^p. \tag{3.1}$$

It is easy to construct a sequence of independent random variables with values in l_1 such that (3.1) does not hold. This implies that l_1 is not B-convex and so it does not have C.A.P. since we have the following result of Bruck [B].

Proposition 3. *A Banach space X has C.A.P. if and only if it is B-convex. Moreover, in this case, there exist constants $c > 0$, $q > 1$ depending on X and r such that, for every $A \subseteq B_r(X)$, we can choose p (in the definition of C.A.P.) via $p \leq c\varepsilon^{q/(1-q)}$.*

The main result is the following:

Theorem 1. *Let X a B-convex Banach space, $h > 0$, and Δ a bounded convex subset of X. Then, for every $f\colon \Delta \to \mathfrak{R}$, which is ε-convex and h-lip, there exists (for ε sufficiently small) a constant K (depending on X, h, and* diam(Δ)*) such that there exists a convex function $g\colon \Delta \to \mathfrak{R}$ satisfying $|f(x) - g(x)| \leq K\varepsilon \lg_2 \varepsilon$.*

We will use the following lemma, the proof of which can be found in several papers and in [C] with the best known constants.

Lemma 1. *Let X be a B-convex Banach space and $f\colon \Delta \subset X \to \Re$ an ε-convex function. Then for $x_0, \dots, x_p \in \Delta$, $\alpha_0, \dots, \alpha_p \in [0, 1]$, $\alpha_1 + \cdots + \alpha_p = 1$, and all integers p, we have $f(\sum_{i=0}^{p} \alpha_i x_i) \le \sum_{i=0}^{p} \alpha_i f(x_i) + 2K_p\varepsilon$. (For the definition of K_n see the introduction.)*

Proof of Theorem 1. Suppose that $\Delta \subseteq B_r(X)$ and let $d = \operatorname{diam}(\Delta)$. We take $Z = X \oplus R$ with $\|z\| = \|(x, y)\| = \|x\| + |y|$ $(x \in X,\ y \in R)$. It is easy to show that Z is a B-convex Banach space since X is B-convex, and so it has C.A.P. Since f is h-lip then f is bounded on Δ. Let $m = \inf_{x\in\Delta} f(x)$, $M = \sup_{x\in\Delta} f(x)$, and $H_0 = \{z \in Z : z = (x, y),\ x \in \Delta,\ f(x) \le y \le M\}$. We have to show that H_0 is a bounded subset of Z. In fact, if we take $z_1, z_2 \in H_0$ we have

$$\|z_1 - z_2\| = \|x_1 - x_2\| + |y_1 - y_2| \le d + M - m;$$

but as a consequence of the lipschitz condition on f we have $M - m \le hd$, so $\operatorname{diam}(H_0) \le (1 + h)d$.

Take $\eta > 0$. Then there exists an integer p such that if $z \in \operatorname{co} H_0$ there exist $z_0, \dots, z_p \in H_0$ and $\alpha_0, \dots, \alpha_p$ so that

$$\left\| z - \sum_{i=0}^{p} \alpha_i z_i \right\| = \left\| x - \sum_{i=0}^{p} \alpha_i x_i \right\| + \left| y - \sum_{i=0}^{p} \alpha_i y_i \right| \le \eta.$$

Then by using Lemma 1

$$\begin{aligned} f(x) &= f(x) - f\left(\sum_{i=0}^{p} \alpha_i x_i\right) + f\left(\sum_{i=0}^{p} \alpha_i x_i\right) \\ &\le \left| f(x) - f\left(\sum_{i=0}^{p} \alpha_i x_i\right)\right| + \sum_{i=0}^{p} \alpha_i f(x_i) + 2K_p\varepsilon \\ &\le h \left\| x - \sum_{i=0}^{p} \alpha_i x_i \right\| + \left| \sum_{i=0}^{p} \alpha_i y_i - y \right| + y + 2K_p\varepsilon \\ &\le h'\eta + y + 2K_p\varepsilon \qquad (h' = \max(h, 1)). \end{aligned}$$

Taking into account the upper bound for p given in Proposition 3 and the value of the constant K_n, we obtain (for p sufficiently large, that is, for η small)

$$f(x) \le h'\eta + (1 + \lg_2 c\eta^{q/(1-q)})\varepsilon + y.$$

Since η was arbitrary, we can choose: $\eta = q\varepsilon/(1 - q)h' \lg_2 e$ (this is the minimum point of the function $g(\eta) = h'\eta + (1 + \lg_2 c\eta^{q/(1-q)})\varepsilon)$. Then we obtain

$$f(x) \le 2K\varepsilon \lg_2 \varepsilon + y \tag{3.2}$$

for some negative constant K. Now define, for $x \in \Delta$, $g_0(x) = \inf\{y \in R : (x, y) \in \operatorname{co} H_0\}$. Then by (3.2) we have $f(x) - 2K\varepsilon \lg_2 \varepsilon \le g_0(x) \le f(x)$. It is not hard to prove that g_0 is a convex function, and if we put $g(x) = g_0(x) + K\varepsilon \lg_2 \varepsilon$ we obtain

$$f(x) - K\varepsilon \lg_2 \varepsilon \le g(x) \le f(x) + K\varepsilon \lg_2,$$

which concludes the proof.

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