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HOMOLOGICAL INVARIANTS OF POWERS OF AN IDEAL

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ABSTRACT. For any fixed nonnegative integer i and all sufficiently large n, the following are shown to be polynomials in n:

(1) The ith Betti number, $\beta_i^{\mathscr{R}_0}(\mathscr{M}_n)$, and the ith Bass number, $\mu^i_{\mathscr{R}_0}(\mathscr{M}_n)$, where $\mathscr{M} = \bigoplus_{n\geq 0}\mathscr{M}_n$ is a finitely generated graded module over a Noetherian graded ring $\mathscr{R} = \mathscr{R}_0[\mathscr{R}_1]$ with $\mathscr{R}_0$ local.

(2) The lengths, $\lambda_R(\mathrm{Tor}_i^R(M/I^nM, Q))$ and $\lambda_R(\mathrm{Ext}^i_R(Q, M/I^nM))$, for an ideal I of a Noetherian ring R and finitely generated R-modules M, Q with $M\otimes_R Q$ of finite length.

(3) The minimal number of generators, $\nu_R(\mathrm{Tor}_i^R(M/I^nM, Q))$ and $\nu_R(\mathrm{Ext}^i_R(Q, M/I^nM))$, where I is an ideal of a Noetherian local ring R and M, Q are finitely generated R-modules.

It is also shown that the degrees of these polynomials are bounded by constants independent of i.

Let R be a Noetherian local ring with maximal ideal $\mathbf{m}$ and residue field k, and let I be a proper ideal of R. Let λ_R and ν_R denote, respectively, the length and minimal number of generator functions on R-modules.

It is well known [9, Theorem 1] that if n is sufficiently large, then $\nu_R(I^n) = \lambda_R(I^n/\mathbf{m}I^n)$ is a polynomial in n. The analytic spread of I, denoted $l(I)$, is defined to be one more than the degree of this polynomial. This result may be rephrased as saying that the first Betti number of R/I^n is a polynomial in n for large n. Recall that for a finitely generated R-module M and nonnegative integer i, the ith Betti number of M, denoted $\beta_i^R(M)$, is defined to be $\lambda_R(\mathrm{Tor}_i^R(M, k))$. Also recall the dual notion of the ith Bass number, defined by $\mu^i_R(M) = \lambda_R(\mathrm{Ext}^i_R(k, M))$.

This paper generalizes the result above to the higher Betti numbers and to Bass numbers. A version for graded modules is also obtained.

Theorem 1. *Let $\mathscr{R} = \bigoplus_{n\geq 0}\mathscr{R}_n$ be a Noetherian graded ring generated as an $\mathscr{R}_0$-algebra by $\mathscr{R}_1$ and with $\mathscr{R}_0$ local with maximal ideal $\mathbf{m}$. Let $\mathscr{M} = \bigoplus_{n\geq 0}\mathscr{M}_n$ be a finitely generated graded $\mathscr{R}$-module. Then, for any fixed integer $i \geq 0$ and for all large n, both $\beta_i^{\mathscr{R}_0}(\mathscr{M}_n)$ and $\mu^i_{\mathscr{R}_0}(\mathscr{M}_n)$ are polynomials in n of degree at most $\dim_{\mathscr{R}}(\mathscr{M}/\mathbf{m}\mathscr{M}) - 1$.*

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The notation in Theorems 2 and 3 is as follows. R is a Noetherian ring, I is an ideal of R, and M, Q are finitely generated R-modules. $\mathcal{R} = \bigoplus_{n\geq 0} I^n$ will denote the Rees ring of I and $\mathcal{M} = \bigoplus_{n\geq 0} I^n M$ the graded $\mathcal{R}$-module associated with the I-adic filtration on M. Let i be any fixed nonnegative integer.

Theorem 2. *If $\lambda_R(M \otimes_R Q) < \infty$, then, for all large n, each of the functions $\lambda_R(\operatorname{Tor}_i^R(M/I^nM, Q))$ and $\lambda_R(\operatorname{Ext}_R^i(Q, M/I^nM))$ is a polynomial in n of degree at most* $\max\{0, \dim_{\mathcal{R}}(\mathcal{M} \otimes_R Q) - 1\}$.

Theorem 3. *If R is a local ring, then, for all large n, each of the functions $\nu_R(\operatorname{Tor}_i^R(M/I^nM, Q))$ and $\nu_R(\operatorname{Ext}_R^i(Q, M/I^nM))$ is a polynomial in n of degree at most* $\max\{0, \dim_{\mathcal{R}}(\mathcal{M} \otimes_R Q \otimes_R k) - 1\}$.

One of the ingredients in the proof of Theorem 1 is the following proposition.

Proposition 4. *Let $\mathcal{R}$ be a Noetherian graded ring, $\mathcal{M}$ a finitely generated graded $\mathcal{R}$-module, and Q a finitely generated $\mathcal{R}_0$-module. Then, for each $i \geq 0$, both $\operatorname{Tor}_i^{\mathcal{R}_0}(\mathcal{M}, Q)$ and $\operatorname{Ext}_{\mathcal{R}_0}^i(Q, \mathcal{M})$ are finitely generated graded $\mathcal{R}$-modules annihilated by some power of* $\operatorname{ann}_{\mathcal{R}}(\mathcal{M} \otimes_{\mathcal{R}_0} Q)$.

Proof. Calculating the Tors and Exts using a resolution of Q by free $\mathcal{R}_0$-modules of finite rank shows that each is a finitely generated graded $\mathcal{R}$-module.

To prove the remark about annihilators, note that $\operatorname{ann}_{\mathcal{R}}(\mathcal{M} \otimes_{\mathcal{R}_0} Q)$ is equal up to radical to $\operatorname{ann}_{\mathcal{R}}(\mathcal{M}) + \operatorname{ann}_{\mathcal{R}}(\mathcal{R} \otimes_{\mathcal{R}_0} Q)$. It is shown by Northcott [8, Lemma 6] that $\operatorname{ann}_{\mathcal{R}}(\mathcal{R} \otimes_{\mathcal{R}_0} Q)$ is equal up to radical to $\operatorname{ann}_{\mathcal{R}_0}(Q) \cdot \mathcal{R}$. Since both $\operatorname{ann}_{\mathcal{R}}(\mathcal{M})$ and $\operatorname{ann}_{\mathcal{R}_0}(Q) \cdot \mathcal{R}$ annihilate each of the Tors and Exts, so does some power of $\operatorname{ann}_{\mathcal{R}}(\mathcal{M} \otimes_{\mathcal{R}_0} Q)$. □

The principal result about the polynomial behaviour of numerical invariants is the Hilbert-Serre theorem, a version of which is stated below.

Theorem (Hilbert-Serre [7, Chapter 7, Theorem 19]). *Let $\mathcal{R} = \bigoplus_{n\geq 0} \mathcal{R}_n$ be a Noetherian graded ring that is generated as an $\mathcal{R}_0$-algebra by $\mathcal{R}_1$. Let $\mathcal{M} = \bigoplus_{n\geq 0} \mathcal{M}_n$ be a finitely generated graded $\mathcal{R}$-module such that, for every $n \geq 0$, the length $\lambda_{\mathcal{R}_0}(\mathcal{M}_n)$ is finite. Then $\lambda_{\mathcal{R}_0}(\mathcal{M}_n)$ is a polynomial in n of degree* $\dim_{\mathcal{R}}(\mathcal{M}) - 1$ *for all large n.*

Proof of Theorem 1. Fix $i \geq 0$ and let k be the residue field of $\mathcal{R}_0$. By Proposition 4, $\operatorname{Tor}_i^{\mathcal{R}_0}(\mathcal{M}, k)$ is a finitely generated graded $\mathcal{R}$-module annihilated by a power of $\operatorname{ann}_{\mathcal{R}}(\mathcal{M}/\mathbf{m}\mathcal{M})$. Applying the Hilbert-Serre theorem shows that $\beta_i^{\mathcal{R}_0}(\mathcal{M}_n) = \lambda_{\mathcal{R}_0}(\operatorname{Tor}_i^{\mathcal{R}_0}(\mathcal{M}_n, k))$ is a polynomial in n of degree at most $\dim_{\mathcal{R}}(\mathcal{M}/\mathbf{m}\mathcal{M}) - 1$ if n is sufficiently large. A similar argument applied to $\operatorname{Ext}_{\mathcal{R}_0}^i(k, \mathcal{M})$ gives the result for $\mu_{\mathcal{R}_0}^i(\mathcal{M}_n)$. □

This theorem can be applied to the Rees ring $\mathcal{R} = \bigoplus_{n\geq 0} I^n$ of an ideal I in a Noetherian local ring R and shows that $\beta_i^R(I^n)$ and $\mu_R^i(I^n)$ are polynomials in n of degree at most $l(I) - 1$ if n is sufficiently large. More generally, if M is a finitely generated R-module, it can be applied to the associated module $\mathcal{M} = \bigoplus_{n\geq 0} I^n M$ which is a finitely generated graded module over the Rees ring $\mathcal{R}$. The conclusion is that, for large n, both $\beta_i^R(I^nM)$ and $\mu_R^i(I^nM)$ are polynomials in n. However, Theorem 1 does not yield information about $\beta_i^R(M/I^nM)$ or about $\mu_R^i(M/I^nM)$ mainly because $\bigoplus_{n\geq 0} M/I^nM$

is not a finitely generated $\mathscr{R}$-module. This difficulty is overcome by the following proposition.

Proposition 5. *Let $\mathscr{R}$ be a Noetherian graded ring and let*

$$(*) \qquad 0 \longrightarrow \mathscr{M} \longrightarrow \mathscr{N} \longrightarrow \mathscr{P} \longrightarrow 0$$

be a short exact sequence of graded $\mathscr{R}$-modules with $\mathscr{M}$ finitely generated. Let Q be a finitely generated $\mathscr{R}_0$-module and i be a fixed nonnegative integer. Then there are induced exact sequences of graded $\mathscr{R}$-modules,

$$0 \longrightarrow \mathscr{K} \longrightarrow \operatorname{Tor}_i^{\mathscr{R}_0}(\mathscr{N}, Q) \longrightarrow \operatorname{Tor}_i^{\mathscr{R}_0}(\mathscr{P}, Q) \longrightarrow \mathscr{C} \longrightarrow 0,$$
$$0 \longrightarrow \mathscr{K}' \longrightarrow \operatorname{Ext}^i_{\mathscr{R}_0}(Q, \mathscr{N}) \longrightarrow \operatorname{Ext}^i_{\mathscr{R}_0}(Q, \mathscr{P}) \longrightarrow \mathscr{C}' \longrightarrow 0,$$

with $\mathscr{K}$, $\mathscr{C}$, $\mathscr{K}'$, and $\mathscr{C}'$ all finitely generated and annihilated by a power of $\operatorname{ann}_{\mathscr{R}}(\mathscr{M} \otimes_{\mathscr{R}_0} Q)$.

Proof. The short exact sequence $(*)$ induces a long exact sequence

$$(**) \qquad \begin{aligned} \cdots &\longrightarrow \operatorname{Tor}_i^{\mathscr{R}_0}(\mathscr{M}, Q) \longrightarrow \operatorname{Tor}_i^{\mathscr{R}_0}(\mathscr{N}, Q) \xrightarrow{p_*} \operatorname{Tor}_i^{\mathscr{R}_0}(\mathscr{P}, Q) \\ &\longrightarrow \operatorname{Tor}_{i-1}^{\mathscr{R}_0}(\mathscr{M}, Q) \xrightarrow{j_*} \cdots . \end{aligned}$$

As in Proposition 4, calculating the Tors using a projective resolution of Q shows that $(**)$ may be regarded as an exact sequence of graded $\mathscr{R}$-modules. Furthermore, $\operatorname{Tor}_i^{\mathscr{R}_0}(\mathscr{M}, Q)$ and $\operatorname{Tor}_{i-1}^{\mathscr{R}_0}(\mathscr{M}, Q)$ are finitely generated graded $\mathscr{R}$-modules annihilated by a power of $\operatorname{ann}_{\mathscr{R}}(\mathscr{M} \otimes_{\mathscr{R}_0} Q)$. Let $\mathscr{K} = \ker(p_*)$ and $\mathscr{C} = \ker(j_*)$ to get the first exact sequence.

Considering the long exact Ext sequence gives $\mathscr{K}'$, $\mathscr{C}'$, and the other exact sequence. □

Proof of Theorem 2. Let $\mathscr{S}$ be the polynomial ring over R in a single variable X and $\mathscr{R}$ be its subring $\{\sum a_i X^i : a_i \in I^i\}$, i.e., $\mathscr{R}$ is the Rees ring of I. $\mathscr{R}$ is a Noetherian graded ring and $\mathscr{S}$ can be regarded as a graded $\mathscr{R}$-module. The kernel of the canonical surjection $\mathscr{S} \otimes_R M \longrightarrow (\mathscr{S}/\mathscr{R}) \otimes_R M$ is the finitely generated graded $\mathscr{R}$-module $\mathscr{M} = \bigoplus_{n \geq 0} I^n M$.

Apply Proposition 5 to the short exact sequence of graded $\mathscr{R}$-modules,

$$0 \longrightarrow \mathscr{M} \longrightarrow \mathscr{S} \otimes_R M \longrightarrow (\mathscr{S}/\mathscr{R}) \otimes_R M \longrightarrow 0.$$

Two exact sequences of graded $\mathscr{R}$-modules are obtained:

$$(\dagger) \quad 0 \longrightarrow \mathscr{K} \longrightarrow \operatorname{Tor}_i^R(\mathscr{S} \otimes_R M, Q) \longrightarrow \operatorname{Tor}_i^R(\mathscr{S}/\mathscr{R} \otimes_R M, Q) \longrightarrow \mathscr{C} \longrightarrow 0,$$
$$(\dagger') \quad 0 \longrightarrow \mathscr{K}' \longrightarrow \operatorname{Ext}^i_R(Q, \mathscr{S} \otimes_R M) \longrightarrow \operatorname{Ext}^i_R(Q, \mathscr{S}/\mathscr{R} \otimes_R M) \longrightarrow \mathscr{C}' \longrightarrow 0,$$

where $\mathscr{K}$, $\mathscr{C}$, $\mathscr{K}'$, and $\mathscr{C}'$ are all finitely generated and annihilated by a power of $\operatorname{ann}_{\mathscr{R}}(\mathscr{M} \otimes_{\mathscr{R}_0} Q)$.

The hypotheses imply that each $M/I^n M \otimes_R Q$ is of finite length and therefore also each $\operatorname{Tor}_i^R(M/I^n M, Q)$. So $(\dagger)$ is a sequence of graded $\mathscr{R}$-modules each of whose homogeneous components are R-modules of finite length. Hence,

$$\lambda_R(\operatorname{Tor}_i^R(M/I^n M, Q)) - \lambda_R(\operatorname{Tor}_i^R(M, Q)) = \lambda_R(\mathscr{C}_n) - \lambda_R(\mathscr{K}_n).$$

By the Hilbert-Serre theorem, the latter is, for large n, a polynomial in n of degree at most $\dim_{\mathscr{R}}(\mathscr{M} \otimes_R Q) - 1$. Since $\lambda_R(\mathrm{Tor}_i^R(M, Q))$ is a constant independent of n, it follows that, for all n sufficiently large, $\lambda_R(\mathrm{Tor}_i^R(M/I^n M, Q))$ is a polynomial in n of degree at most $\max\{0, \dim_{\mathscr{R}}(\mathscr{M} \otimes_R Q) - 1\}$.

The result for $\lambda_R(\mathrm{Ext}_R^i(Q, M/I^n M))$ uses (†′) and follows by a dual analysis. □

Proposition 6. *Let $\mathscr{R}$ be a Noetherian graded ring generated by $\mathscr{R}_1$ as an $\mathscr{R}_0$-algebra and with $\mathscr{R}_0$ local. Let*

$$0 \longrightarrow \mathscr{K} \longrightarrow \mathscr{N} \longrightarrow \mathscr{P} \longrightarrow \mathscr{C} \longrightarrow 0$$

be an exact sequence of graded $\mathscr{R}$-modules where $\mathscr{K}$ and $\mathscr{C}$ are finitely generated. Then, if n is sufficiently large, $\nu_{\mathscr{R}_0}(\mathscr{P}_n) - \nu_{\mathscr{R}_0}(\mathscr{N}_n)$ is a polynomial in n. The degree of this polynomial is at most $\dim_{\mathscr{R}}(\mathscr{R}/\mathscr{I}) - 1$, where $\mathscr{I} = \mathbf{m}\mathscr{R} + (\mathrm{ann}_{\mathscr{R}}(\mathscr{K}) \cap \mathrm{ann}_{\mathscr{R}}(\mathscr{C}))$.

Proof. Break up the exact sequence into short exact sequences

$$0 \longrightarrow \mathscr{K} \longrightarrow \mathscr{N} \longrightarrow \mathscr{H} \longrightarrow 0,$$
$$0 \longrightarrow \mathscr{H} \longrightarrow \mathscr{P} \longrightarrow \mathscr{C} \longrightarrow 0.$$

Tensor these sequences over $\mathscr{R}_0$ with the residue field, k, of $\mathscr{R}_0$. This gives the exact sequences

$$\mathscr{K} \otimes_{\mathscr{R}_0} k \longrightarrow \mathscr{N} \otimes_{\mathscr{R}_0} k \longrightarrow \mathscr{H} \otimes_{\mathscr{R}_0} k \longrightarrow 0,$$
$$\mathrm{Tor}_1^{\mathscr{R}_0}(\mathscr{C}, k) \longrightarrow \mathscr{H} \otimes_{\mathscr{R}_0} k \longrightarrow \mathscr{P} \otimes_{\mathscr{R}_0} k \longrightarrow \mathscr{C} \otimes_{\mathscr{R}_0} k \longrightarrow 0.$$

Proposition 4 shows that $\mathscr{K} \otimes_{\mathscr{R}_0} k$, $\mathscr{C} \otimes_{\mathscr{R}_0} k$, and $\mathrm{Tor}_1^{\mathscr{R}_0}(\mathscr{C}, k)$ are all finitely generated $\mathscr{R}$-modules that are annihilated by $\mathscr{I}$. Then using the additivity of length and the Hilbert-Serre theorem completes the proof. □

Proof of Theorem 3. With notation as in the proof of Theorem 2, consider the exact sequence obtained therein,

$$(\dagger') \quad 0 \longrightarrow \mathscr{K}' \longrightarrow \mathrm{Ext}_R^i(Q, \mathscr{S} \otimes_R M) \longrightarrow \mathrm{Ext}_R^i(Q, \mathscr{S}/\mathscr{R} \otimes_R M) \longrightarrow \mathscr{C}' \longrightarrow 0,$$

with $\mathscr{K}'$ and $\mathscr{C}'$ finitely generated $\mathscr{R}$-modules annihilated by a power of $\mathrm{ann}_{\mathscr{R}}(\mathscr{M} \otimes_R Q)$. By applying Proposition 6, it follows that, for large n, $\nu_R(\mathrm{Ext}_R^i(Q, M/I^n M)) - \nu_R(\mathrm{Ext}_R^i(Q, M))$ is a polynomial in n of degree at most $\dim_{\mathscr{R}}(\mathscr{M} \otimes_R Q \otimes_R k) - 1$. So if n is large, $\nu_R(\mathrm{Ext}_R^i(Q, M/I^n M))$ is a polynomial in n of degree at most $\max\{0, \dim_{\mathscr{R}}(\mathscr{M} \otimes_R Q \otimes_R k) - 1\}$.

As usual, a dual proof works for the Tors. □

In the following corollaries, R will denote a Noetherian local ring with maximal ideal $\mathbf{m}$ and residue field k, and I will denote a proper ideal of R. The analytic spread, $l(I)$, of I is equal to $\dim(\mathscr{R}/\mathbf{m}\mathscr{R})$, where $\mathscr{R}$ is the Rees ring of I by [9, Theorem 2]. Let $M \neq 0$ be a finitely generated R-module. Define the analytic spread of I on M to be $l_M(I) = \dim_{\mathscr{R}}(\mathscr{M}/\mathbf{m}\mathscr{M})$, where $\mathscr{M} = \bigoplus_{n \geq 0} I^n M$ is the graded $\mathscr{R}$-module asssociated with the I-adic filtration on M. Note that $l_M(I)$ is bounded above by $l(I)$ for any M.

Corollary 7. *For any fixed nonnegative integer i and for all n sufficiently large, $\beta_i^R(M/I^nM)$ and $\mu_R^i(M/I^nM)$ are polynomials in n of degree at most* $\max\{0, l_M(I)-1\}$.

Proof. Apply Theorem 2 with $Q = k$. □

The result in Corollary 7 about $\mu_R^i(M/I^nM)$ answers a question of Professor Bernard Johnston. Corollary 8 is essentially due to Mark Johnson.

Corollary 8. *For all large n, both $\mathrm{pd}_R(M/I^nM)$ and $\mathrm{id}_R(M/I^nM)$ attain stable constant values (possibly infinite).*

Proof. Let $d = \dim(R)$. Consider $\beta_{d+1}^R(M/I^nM)$. If this is a nonzero polynomial for all large n, then $\mathrm{pd}_R(M/I^nM) \geq d+1$ for such n. Hence, by the Auslander-Buchsbaum theorem, $\mathrm{pd}_R(M/I^nM)$ is infinite for all large n. Otherwise, let p be the largest i for which $\beta_i^R(M/I^nM)$ is a nonzero polynomial for all large n. Then p is at most d; in this case, $\mathrm{pd}_R(M/I^nM)$ has a stable value of p for all large n.

A dual proof shows stability of $\mathrm{id}_R(M/I^nM)$. □

Remark. It is shown by Avramov [1, Appendix] that, for all large n and for every $i \geq 0$, there is an exact sequence

$$0 \longrightarrow \mathrm{Tor}_i^R(M, k) \longrightarrow \mathrm{Tor}_i^R(M/I^nM, k) \longrightarrow \mathrm{Tor}_{i-1}^R(I^nM, k) \longrightarrow 0.$$

This implies that, for all n sufficiently large,

$$\beta_i^R(M/I^nM) = \beta_i^R(M) + \beta_{i-1}^R(I^nM).$$

In particular, if M/I^nM has finite projective dimension for all large n, then M itself is of finite projective dimension and so are all I^nM for large n.

As another corollary, a new proof of a special case of a theorem due to Brodmann [2, Theorem (2)(i)] is obtained. Brodmann's result applies to the case where R is not assumed local. I have been unable to adapt my proof to cover this case.

Corollary 9. *If J is a proper ideal of R, then, for all large n, $\mathrm{depth}_J(M/I^nM)$ attains a stable constant value.*

Proof. Theorem 3 applied with $Q = R/J$ implies that $\nu_R(\mathrm{Ext}_R^i(R/J, M/I^nM))$ is a polynomial in n for all large n. Let p be the least integer i for which $\nu_R(\mathrm{Ext}_R^i(R/J, M/I^nM))$ is not the zero polynomial for all large n. Since $M \neq 0$ and I, J are proper ideals, p exists and is at most $d = \dim(R)$.

For all large n, $\mathrm{Ext}_R^p(R/J, M/I^nM) \neq 0$ but $\mathrm{Ext}_R^i(R/J, M/I^nM) = 0$ for $i < p$. Since $\mathrm{depth}_J(M/I^nM) = \min\{i : \mathrm{Ext}_R^i(R/J, M/I^nM) \neq 0\}$, it has a stable value of p for all large n. □

Example 10. Let $a_1, a_2, \ldots, a_k$ be a regular sequence in a Noetherian local ring R and let I be the ideal they generate. Then for any $n > 0$, I^n is generated by the maximal minors of the $n \times (n+k-1)$ matrix A where

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_k & 0 & 0 & \cdots & 0 \\ 0 & a_1 & a_2 & \cdots & a_{k-1} & a_k & 0 & \cdots & 0 \\ 0 & 0 & a_1 & \cdots & a_{k-2} & a_{k-1} & a_k & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 & a_2 & a_3 & \cdots & a_k \end{pmatrix}.$$

Furthermore, $\operatorname{grade}_R(I^n) = k = (n+k-1) - n + 1$. So by the results of Eagon-Northcott [3, Theorem 2], their complex E^A associated to the matrix A resolves R/I^n minimally. The Betti numbers of R/I^n can therefore be read off as

$$\beta_i^R(R/I^n) = \binom{n+k-1}{k-i}\binom{n+i-2}{i-1}, \qquad 1 \le i \le k.$$

These are polynomials in n of degree $k - 1 = l(I) - 1$ for all values of n.

Example 11. Let R be the localization at its maximal graded ideal of a polynomial ring over a field in the entries, X_{ij}, of a generic $(k+1) \times k$ matrix. Let I be the ideal generated by the maximal minors of the matrix. Then, it is known [5, §4] that the kernel of a presentation $R[T_1, \dots, T_{k+1}] \longrightarrow \mathscr{R}$ of the Rees ring of I is given by the ideal

$$K = (X_{11}T_1 + \cdots + X_{k+1,1}T_{k+1}, \dots, X_{k1}T_1 + \cdots + X_{k+1,k}T_{k+1}),$$

which is a complete intersection.

Hence the Koszul complex for these generators is a resolution of $\mathscr{R}$ over $R[T_1, \dots, T_{k+1}]$. In any particular degree n, an R-free resolution of I^n is obtained. Since $K \subseteq (X_{ij})R[T_1, \dots, T_{k+1}]$, each of these resolutions is seen to be minimal.

Thus the Betti numbers of I^n and hence also of R/I^n can be explicitly computed from this complex as

$$\beta_i^R(R/I^n) = \binom{k}{i-1}\binom{n+k-i+1}{k}, \qquad 1 \le i \le k+1.$$

The degrees of all these polynomials are equal to $k = l(I) - 1$. Note that $\operatorname{depth}_R(R/I^n)$ decreases from $k^2 + k - 2$ to $k^2 - 1$ as n increases from 1 to k and then remains $k^2 - 1$ for all larger values of n.

Example 12. Let R be a Noetherian local ring with maximal ideal $\mathbf{m}$ and residue field k, and let M be a finitely generated R-module of dimension d. Then, for each $i \ge 0$, the polynomial that gives $\beta_i^R(\mathbf{m}^n M)$ for large n either vanishes or is of degree $d - 1$. The proof is dealt with in three cases.

If R is 0-dimensional, then it is clear that $\beta_i^R(\mathbf{m}^n M)$ vanishes for each $i \ge 0$ and all sufficiently large n.

The proof in the remaining cases depends on a formula of Levin [6, Proof of Theorem 2] which shows that, for a fixed $i \ge 0$ and all sufficiently large n, there is an equality

$$\beta_i^R(\mathbf{m}^n M) = h_n b_i - h_{n+1} b_{i-1} + \cdots + (-1)^i h_{n+i} b_0,$$

where $h_n = \lambda_R(\mathbf{m}^n M / \mathbf{m}^{n+1} M)$ and b_j is the jth Betti number of k. Since h_n is a polynomial in n of degree $d - 1$ for large n, this shows that the same is true of $\beta_i^R(\mathbf{m}^n M)$ unless $b_i - b_{i-1} + \cdots + (-1)^i b_0$ vanishes.

Next, suppose that R is a domain. Then $b_i - b_{i-1} + \cdots + (-1)^i b_0$ is the rank of the $(i+1)$st syzygy of k over the field of fractions of R and its vanishing implies that R is in fact a regular local ring of dimension at most i. Using the values of the Betti numbers of k and Levin's formula, $\beta_i^R(\mathbf{m}^n M)$ is, for large n, a difference quotient of h_n of order at least d. Since h_n is a polynomial of degree $d - 1$, it follows that $\beta_i^R(\mathbf{m}^n M)$ vanishes for large n.

If neither of the above hold then R is a nonregular ring of embedding dimension at least 2. By a theorem of Gulliksen [4, Corollaries 1 and 2] there is an equality of formal power series

$$b_0 + b_1 t + b_2 t^2 + \cdots = \frac{(1+t)^{e_1}(1+t^3)^{e_3}\cdots}{(1-t^2)^{e_2}(1-t^4)^{e_4}\cdots},$$

where $e_i \geq 0$. Furthermore, e_1 is the embedding dimension of R, which in this case is at least 2, and $e_2 > 0$ as R is not regular.

The alternating sum $b_i - b_{i-1} + \cdots + (-1)^i b_0$ is the coefficient of t^i in

$$\frac{(1-(-1)^{i+1}t^{i+1})}{(1+t)} \frac{(1+t)^{e_1}(1+t^3)^{e_3}\cdots}{(1-t^2)^{e_2}(1-t^4)^{e_4}\cdots}.$$

The power series above is coefficientwise greater than or equal to

$$\frac{(1-(-1)^{i+1}t^{i+1})}{(1-t)}$$

since what is left out is a product of power series with nonnegative coefficients.

Finally, the coefficient of t^i in the last series is 1 and therefore $b_i - b_{i-1} + \cdots + (-1)^i b_0 \geq 1$.

The proof of positivity of $b_i - b_{i-1} + \cdots + (-1)^i b_0$ is due to Professor Avramov.

Question 13. For an ideal I in a Noetherian local ring R, a finitely generated R-module M, and any $i \geq 0$, is it true that the polynomial that gives $\beta_i^R(I^n M)$ for large n either vanishes or has degree $l_M(I) - 1$?

I do not know the answer even when $M = R$ which is assumed to be a regular local ring and I is an arbitrary ideal primary to the maximal ideal of R.

Added in proof

Professor Herzog has shown me a proof that Question 13 has an affirmative answer over a regular local ring if the Rees ring of the ideal is Cohen-Macaulay.

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