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FIXED POINTS OF FINITE GROUPS OF FREE GROUP AUTOMORPHISMS

TAD WHITE

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ABSTRACT. We construct an equivariant collection of contractions of the compactified Culler-Vogtmann outer space $\overline{X}_n$. As a consequence, we prove that any finite subgroup of the outer automorphism group of a free group fixes a contractible subset of $\overline{X}_n$.

1. INTRODUCTION

Culler and Vogtmann [CV] initiated a study of the outer automorphism group $\mathrm{Out}(F_n)$ of the free group F_n on n letters by constructing a space X_n upon which $\mathrm{Out}(F_n)$ acts properly discontinuously. This space, which has since come to be known as the "outer space", consists of projective classes of free minimal actions of F_n on simplicial metric trees, where two such trees are identified if they are equivariantly isometric. The quotient of each such tree by such an action is a finite marked metric graph in which each vertex has valence ≥ 3. Here, "metric" means that each edge has a length, and a "marking" is a preferred homotopy equivalence of the wedge R_n of n circles to this graph. One thus has two possible views of X_n; these are interchangeable. The group $\mathrm{Out}(F_n)$ acts in the obvious way: one represents an automorphism of F_n as a self-map of R_n and precomposes the marking with this map.

Culler and Vogtmann showed that X_n is a finite-dimensional contractible space and that the $\mathrm{Out}(F_n)$-action has finite stabilizers and finite quotient. Furthermore, X_n has a natural compactification $\overline{X}_n$ in which the boundary consists of certain actions of F_n on $\mathbf{R}$-trees with cyclic edge stabilizers. It is known that $\overline{X}_n$ is contractible; this was first shown by Steiner [St] and, independently, by Skora [Sk]. Skora's technique was to construct a path between two $\mathbf{R}$-trees, given a morphism between them. (A *morphism* is a map such that each segment in the domain $\mathbf{R}$-tree contains an initial arc which is mapped isometrically.) In this paper we generalize Skora's methods to prove the following theorem:

Theorem. *Let $\mathscr{X}$ be the space of nontrivial semisimple actions of F_n on $\mathbf{R}$-trees. Then there is a continuous deformation $F : X_n \times \mathscr{X} \times I \to \mathscr{X}$ such that*

(1) $F(T_0, T_1, 0) = T_0$;

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(2) $F(T_0, T_1, 1) = T_1$;
(3) $F(T_0, T_0, t) = T_0$ *for all* t; *and*
(4) F *is equivariant under the diagonal action of* $\mathrm{Out}(F_n)$ *on* $X_n \times \mathscr{X}$.

The result also holds if we replace $\mathscr{X}$ by the compactification $\overline{X}_n \subset \mathscr{X}$ of X_n. An immediate consequence is the following:

Corollary. *The subset* X_G *(resp.* $\overline{X}_G$*) of* X_n *(resp.* $\overline{X}_n$*) fixed by a finite subgroup* G *of* $\mathrm{Out}(F_n)$ *is contractible.*

2. Preliminaries

Group actions on R-trees. An **R**-tree is a metric space such that any two points are joined by a unique embedded arc, which is isometric to a finite interval in the real line. For a general exposition of the theory of group actions on **R**-trees, see, for example, [CM].

Let $G \times T \to T$ be an action of a finitely generated group G on an **R**-tree. The metric on T will be denoted by d or d_T as the context requires. The *translation length function* $l(g)$ determined by the action is the map $l : G \to \mathbf{R}$ defined by $l(g) = \min_{x \in T} d(x, gx)$. The *characteristic set* of g is $T_\gamma = \{x \in T \mid d(x, gx) = l(g)\}$. Since $l(g) = l(xgx^{-1})$ for all $g, x \in G$, we may consider l as a real-valued function on the set C of conjugacy classes in G. Actions of G on **R**-trees T and T' are *projectively equivalent* if there is an equivariant homeomorphism from T to T' such that the induced pullback metric on T differs from the original metric by a nonzero multiplicative constant.

An action is *irreducible* if no finite set of ends of T is invariant under G and *dihedral* if G interchanges a pair of ends. The action is *semisimple* if it is irreducible, dihedral, or trivial.

We now describe a topology, due to Paulin [P], on the space of actions of G on **R**-trees. An *ε-approximation* between two compact metric spaces K and L is a subdirect product $R \subset K \times L$ (i.e., the relation R surjects onto both K and L) such that if $x_0 R y_0$ and $x_1 R y_1$, then $|d(x_0, x_1) - d(y_0, y_1)| < \varepsilon$. Following Gromov [G], define $d_G(K, L) < \varepsilon$ if there exists an ε-approximation between K and L. More generally, suppose X is a G-metric space (i.e., a metric space with an associated G-action) and that P is a finite subset of G. An ε-approximation is *P-equivariant* if whenever $g \in P$, $x \in K$, $gx \in K$, $y \in L$, and xRy, then $gy \in L$ and $gxRgy$.

Now fix a compact $K \subset X$, a finite subset P of G, and $\varepsilon > 0$. Paulin [P] defines the basic neighborhood $U(X, K, P, \varepsilon)$ to be the collection of G-metric spaces Y such that there exists a compact $L \subset Y$ and a P-equivariant ε-approximation from K to L and uses these as a neighborhood base for a topology on G-metric spaces.

Skora [Sk] generalizes this to define a topology on a space $\mathscr{C}$ of equivariant maps between G-metric spaces. We give the definitions here for completeness. If $\phi : X \to X'$ and $\psi : Y \to Y'$ are maps, an *ε-approximation* from ϕ to ψ is a pair of relations (R, R') such that

(1) R and R' are ε-approximations from X to Y and from X' to Y' respectively; and
(2) xRy implies $\phi(x) R' \psi(y)$.

We endow a product $X \times X'$ with the uniform metric:

$$d((x_0, x_0'), (x_1, x_1')) = \max\{d(x_0, x_1), d(x_0', x_1')\}.$$

Given ε-approximations R from X to Y and R' from X' to Y', one sees that (R, R') is an ε-approximation from $X \times X'$ to $Y \times Y'$, where (R, R') is the product of R and R' as subsets of $X \times Y$ and $X' \times Y'$ respectively.

The notion of P-equivariance extends naturally to the case of ε-approximations between maps. Skora defines a topology on a space $\mathscr{C}$ of equivariant maps between G-metric spaces as follows: Let $K \times K'$ be a compact subset of $X \times X'$, P a finite subset of G, and $\varepsilon > 0$. The basic neighborhood $U(\phi, K \times K', P, \varepsilon)$ of ϕ is the set of all maps $\psi : Y \to Y'$ in $\mathscr{C}$ such that for some compact $L \times L' \subseteq Y \times Y'$, there is a P-equivariant, closed ε-approximation from $\phi|K$ to $\psi|L$. $\mathscr{C}$ is then given the topology generated by these basic open sets.

A universal bundle. Fix a group G and a space $\mathscr{X}$ of actions of G on $\mathbf{R}$-trees. In this section we introduce a bundle over $\mathscr{X}$ which should be thought of as an analogue of the universal Teichmüller bundle over Teichmüller space. Although it is essentially a notational convenience, this bundle will be useful in describing some constructions later and in dealing with the assorted group actions which arise.

Definition 2.1. Let $\mathscr{X}$ be a space of G-actions on $\mathbf{R}$-trees. The *tree bundle* $\mathscr{T}(\mathscr{X})$ over $\mathscr{X}$ is the space $\{(T, x) \mid T \in \mathscr{X} \text{ and } x \in T\}$. If the $\mathbf{R}$-trees are simplicial, we further define the *vertex bundle* $\mathscr{V}(\mathscr{X})$ over $\mathscr{X}$ to be the space $\{(T, v) \mid T \in \mathscr{X} \text{ and } v \text{ is a vertex of } T\}$. There is a natural G-action on each fiber of these bundles.

$\mathrm{Aut}(G)$ acts naturally on $\mathscr{T}(\mathscr{X})$ by permuting the fibers. If we fix a group action $G \times T \to T$ in $\mathscr{X}$ and $\phi \in \mathrm{Aut}(G)$, we can define the image of this action under ϕ by precomposing:

$$G \times T \xrightarrow{\gamma} T \mapsto G \times T \xrightarrow{\gamma\circ(\phi\times\mathrm{id})} T.$$

In $\mathscr{T}(\mathscr{X})$, then, we define $\phi(T, x) = (T', x')$ if there exists an isometry of pointed spaces $f : (T, x) \to (T', x')$ which induces ϕ in the sense that the following diagram commutes:

$$\begin{tikzcd} G \times T \arrow[r] \arrow[d, "\phi\times f"'] & T \arrow[d, "f"] \\ G \times T' \arrow[r] & T' \end{tikzcd}$$

This action restricts to an action on $\mathscr{V}(\mathscr{X})$. Note that the "vertical" action of G on the fibers is simply the restriction of the $\mathrm{Aut}(G)$ action to $\mathrm{Inn}(G)$, and since $\mathrm{Inn}(G)$ acts trivially on the base space, we recover the standard $\mathrm{Out}(G)$ action on $\mathscr{X}$.

Finally, there is a natural topology on the tree bundle.

T and T' are close if there exists an approximating relation R between large compact subsets of T and T', such that R is equivariant with respect to a large finite subset of G. Hence two points (T, x) and (T', x') in the bundle are close if xRx' for such a relation R. We endow the vertex bundle with the topology it inherits from the tree bundle.

3. Weighted length functions

Throughout this section, G denotes a finitely generated group. All actions of G are assumed to be minimal.

We now generalize the notions of translation length function and characteristic set. Let G act on T, and let λ be a real-valued function on G. The *weighted length function* $f_\lambda : T \to \mathbf{R}$ is defined to be $\sup_{g\in G} \lambda(g)d(x, gx)$. We define the *length* of λ to be the infimum of f_λ: $l(\lambda) = l_T(\lambda) = \inf_{x\in T} f_\lambda(x)$. (We will omit the subscript if the tree is clear from the context.) The *characteristic set* of λ is $T_\lambda = \{ x \in T \mid f_\lambda(x) = l(\lambda) \}$.

Remarks. 1. For a general function λ, it may happen that $l(\lambda) = \infty$ and that $T_\lambda = T$. We will eventually restrict our attention to functions λ for which this does not occur, but remark that the definitions still make sense in this case.

2. Skora [Sk] considers the case in which λ is the characteristic function of a finite generating set of G. The arguments here have a slightly different flavor, since the weighting functions now have infinite support.

3. The standard translation length $l(g)$ of a group element g is recovered in this context by taking λ to be the function taking the value 1 at g and 0 elsewhere. Just as $l(g)$ is invariant under conjugacy in G, the length $l(\lambda)$ is invariant under the "inner" action of G on $\mathbf{R}^G$ by conjugation.

There is a natural action of $\operatorname{Aut}(G)$ on $\mathbf{R}^G$: given $\lambda \in \mathbf{R}^G$ and an automorphism ϕ of G, one can define $\phi(\lambda) = \lambda \circ \phi^{-1}$. The following lemma is immediate.

Lemma 3.1. *If $\phi \in \operatorname{Aut}(G)$, $\lambda \in \mathbf{R}^G$, and f is an isometry from (T, v) to (T', v') inducing ϕ as in the preceding section, then:*

(1) $l_{f(T)}(\lambda) = l_T(\phi(\lambda))$, *and*

(2) $T_{\phi(\lambda)} = f(T_\lambda)$.

In other words, the generalized notions of length function and characteristic set are equivariant. □

For the remainder of this section, we shall assume G acts freely on a simplicial $\mathbf{R}$-tree T (and hence $G = F_n$). We now choose, for each point of T, a weighting function on G:

Definition. Let $G \times T \to T$ be a free simplicial action. For any $x \in T$, define a weighting function $\lambda_{T,x}: G \to \mathbf{R}$ by $\lambda_{T,x}(g) = 1/d_T(x, gx)$ if $g \neq \mathrm{id}$, and $\lambda_{T,x}(\mathrm{id}) = 0$.

Then, for any semisimple action $G \times T' \to T'$, we let $f_{T,x}: T \to \mathbf{R}$ denote the weighted length function corresponding to $\lambda_{T,x}$:

$$f_{T,x}(y) = \sup_{g\in G} \lambda_{T,x}(g)d(y, gy) = \sup_{g\in G-\{\mathrm{id}\}} \frac{d_{T'}(y, gy)}{d_T(x, gx)}.$$

We first analyze the behavior of the function $f_{T,x}$ on T. A priori, the supremum defining $f_{T,x}$ may not even be finite, so this is the first issue to be dealt with.

Lemma 3.2. *Suppose that G acts irreducibly on T. Then for each $x \in T$, the function $f_{T,x} : T \to \mathbf{R}$ is:*

(1) *finite*;
(2) *proper*; *and*
(3) *has a unique minimum at x.*

Proof. (1) Fix $y \in T$. Then for any $g \in G$,

$$d(y, gy) \leq d(x, y) + d(x, gx) + d(gx, gy) = d(x, gx) + 2d(x, y),$$

so that $\lambda_{T,x}(g)d(y, gy) \leq 1 + 2d(x, y)/d(x, gx)$. But the action on T is free and simplicial, so $d(x, gx)$ is uniformly bounded away from zero for all $g \neq \mathrm{id}$. Thus $\lambda_{T,x}(g)d(y, gy)$ is uniformly bounded in g, and thus $f_{T,x}(y) < \infty$.

(2) Let y be any point in T other than x. By the minimality of T, we can choose an element g of G such that x lies on the segment joining y to the axis of G. For such a g, $d(y, gy) = d(x, gx) + 2d(x, y)$. Hence $f_{T,x}(y)$ increases with $d(x, y)$. This also proves (3). □

We shall require the following lemma:

Lemma 3.3. *Let $\lambda: G \to \mathbf{R}$ be a real-valued function. Suppose that $G \times T \to T$ is a free simplicial action such that the weighted length function f_λ is finite and proper on T. Then f_λ is finite and proper for any semisimple action of G on an $\mathbf{R}$-tree T'.*

Proof. Let $h : T \to T'$ be an edgewise-linear equivariant map. We use h to relate the behavior of f_λ on T' to its behavior on T. Since T has compact quotient, h is L-Lipschitz for some $L > 0$. Furthermore, h is onto, since the action on T' is minimal. Let $z \in T$ be an h-preimage of $y \in T'$. Then $d_{T'}(y, gy) \leq Ld_T(z, gz)$ for any $g \in G$. Thus $f_\lambda(y) \leq Lf_\lambda(z)$. By the preceding lemma, f_λ is finite on T, so it is finite on T'. That f_λ is proper follows as in the preceding lemma. □

Definition 3.4. Let Λ denote the set of real-valued functions λ on G such that the weighted length function $f_\lambda(x)$ is a proper, real-valued function for a free action of G on a simplicial $\mathbf{R}$-tree T. Since any two such actions are equivariantly quasi-isometric, the preceding lemma implies that Λ is independent of the choice of T. We remark that the weighted length function is automatically convex for each $\lambda \in \Lambda$, as each function $d_T(x, gx)$ is convex.

Remark. The correspondence $(T, x) \mapsto \lambda_{T,x}$ gives a continuous map from the tree bundle $\mathcal{T}(X_n)$ to $\mathbf{R}^G$. Lemma 3.2 thus says that the image of this map is contained in Λ.

Now suppose $\lambda \in \Lambda$, and let T be an $\mathbf{R}$-tree equipped with a semi-simple action of G. Since f_λ is convex and proper on T, its characteristic set T_λ is a nonempty compact subtree of T. We can then choose a "base point" $b(T, \lambda)$ in T by selecting the "center" of T_λ; that is, the unique point y such that the ball at y of radius $(\operatorname{diam} T_\lambda)/2$ is contained in T_λ. Hence $b(\cdot, \lambda)$ may be thought of as a section of the tree bundle $\mathcal{T}(\mathcal{X})$. The following shows that these sections are continuous and vary continuously with λ.

Proposition 3.5. *Let $G = F_n$ and Λ be as above. Let $\mathcal{X}$ denote the space of semisimple actions of G on $\mathbf{R}$-trees and $\mathcal{T}(\mathcal{X})$ be the tree bundle over $\mathcal{X}$. Then the associated base point function $b: \mathcal{X} \times \Lambda \to \mathcal{T}(\mathcal{X})$ is continuous.*

The proof is essentially identical to that of Proposition 5.2 in [Sk]. More generally, if G is a finitely generated group, $\mathscr{X}$ a space of actions of G on $\mathbf{R}$-trees, and Λ a collection of weight functions on G such that f_λ is finite and proper for each $\lambda \in \Lambda$ and each action in $\mathscr{X}$, then the same result holds.

Lemma 3.6. *b is equivariant under the action of* $\mathrm{Aut}(F_n)$ *on* $\mathscr{X} \times \Lambda$.

Proof. This follows immediately from Lemma 3.1. □

4. Paths between R-trees

In this section we take G to be the free group F_n of a fixed rank n. Recall [CV] that "outer space" is the space X_n of projective classes of minimal free actions of F_n on simplicial $\mathbf{R}$-trees. A map from a simplicial $\mathbf{R}$-tree to an arbitrary $\mathbf{R}$-tree is *transverse* if it is linear on each 1-simplex.

If f is any function, we will denote by $\mathscr{D}(f)$ and $\mathscr{R}(f)$ the domain and range of f.

Proposition 4.1. *Let $\mathscr{X}$ be a space of nontrivial irreducible actions of F_n on $\mathbf{R}$-trees, and let $\mathscr{C} = \mathscr{C}(X_n, \mathscr{X})$ be the space of all equivariant transverse maps from simplicial $\mathbf{R}$-trees to elements of $\mathscr{X}$. Then there is a map $B\colon X_n \times \mathscr{X} \to \mathscr{C}$ such that $\mathscr{D}(B(Y, T)) = Y$, $\mathscr{R}(B(Y, T)) = T$, and $B(Y, Y) = \mathrm{Id}_Y$ for all $Y \in X_n \cap \mathscr{X}$, $T \in \mathscr{X}$. Furthermore, B is equivariant with respect to the diagonal action of* $\mathrm{Out}(F_n)$ *on $X_n \times \mathscr{X}$.*

Proof. Given $Y \in X_n$ and $T \in \mathscr{X}$, we use the base point map to define the image of a vertex v of Y: $B(Y, T)(v) = b(T, \lambda_{Y,v})$. Now extend the domain of $B(Y, T)$ to all of Y by mapping linearly on the edges of Y. Now $\lambda_{Y,v}$ is continuous, so $B(Y, T)$ is continuous by Lemma 3.3 and Proposition 3.5. $B(Y, T)$ is equivariant under $\mathrm{Aut}(F_n)$ by Lemma 3.6. Finally, by Lemma 3.2, the weighted length function $f_{Y,v}$ on Y has a unique minimum at v, so $B(Y, Y)$ is the identity on the vertices of Y and hence on all of Y by linearity. □

Remark. In the above proposition, it is essential that the action on the source $\mathbf{R}$-tree be free. If any $g \in F_n$ has a fixed point in Y, it is impossible in general to construct an equivariant map from Y to T.

Recall [Sk] that an (equivariant) *morphism* between $\mathbf{R}$-trees is an (equivariant) map such that each segment in the source $\mathbf{R}$-tree has a nontrivial initial segment which is mapped isometrically onto its image. Given a surjective morphism $\phi : T_0 \to T_1$, Skora constructs morphisms ϕ_{st} for $0 \le s \le t \le 1$ such that $\phi_{01} = \phi$, $\phi_{st} \circ \phi_{rs} = \phi_{rt}$, and $\phi_{rs} = \mathrm{id}$ whenever $\phi = \mathrm{id}$. We will need the following result:

Proposition 4.2 [Sk, 4.8]. *Let G be a group, $\mathscr{X}$ the space of all actions of G on $\mathbf{R}$-trees, and $\mathscr{C}(\mathscr{X})$ the space of all morphisms between elements of $\mathscr{X}$. Then the function*

$$\mathscr{C}(\mathscr{X}) \times \{(s, t) \mid 0 \le s \le t \le 1\} \to \mathscr{C}(\mathscr{X})$$

defined via $(\phi, (s, t)) \mapsto \phi_{st}$ is continuous. Furthermore, this map is equivariant with respect to the natural action of $\mathrm{Out}(F_n)$ *on $\mathscr{C}(\mathscr{X})$.*

Hence a morphism between two $\mathbf{R}$-trees determines a canonical path between them. We now improve Proposition 4.1 to promote $B(Y, T)$ to a morphism:

Proposition 4.3. *Let $\mathscr{X}$ be the space of nontrivial semisimple actions of F_n on* **R**-*trees, and let $\mathscr{C} = \mathscr{C}(\overline{X}_n, \mathscr{X})$ be the space of all equivariant transverse morphisms from elements of $\overline{X}_n$ to elements of $\mathscr{X}$. Then there is a map $M : X_n \times \mathscr{X} \to \mathscr{C}$ such that the following hold for all $Y \in X_n$, $T \in \mathscr{X}$:*

(1) $\mathscr{R}(M(Y, T)) = T$;
(2) $\mathscr{D}(M(Y, T))$ *is a simplicial* **R**-*tree contained in the closure of the simplex of X_n containing Y;*
(3) *if T is a free simplicial action, so is $\mathscr{D}(M(Y, T))$;*
(4) $M(Y, Y) = \mathrm{Id}_Y$; *and*
(5) *M is equivariant under the diagonal action of* $\mathrm{Out}(F_n)$.

Proof. For each 1-simplex σ in Y, $B(Y, T)$ maps σ linearly into T, dilating by some nonnegative number μ. Multiply the metric on σ by μ. Let Y' be the tree obtained from Y in this way. $B(Y, T)$ then induces a morphism $M(Y, T)$ from Y' to T. M clearly varies continuously with Y and T, and the conclusions all follow immediately from the corresponding properties of B. □

Theorem 4.4. *There is a continuous function $F\colon X_n \times \overline{X}_n \times I \to \overline{X}_n$ such that*

(1) $F(T_0, T_1, 0) = T_0$;
(2) $F(T_0, T_1, 1) = T_1$;
(3) $F(T_0, T_0, t) = T_0$ *for all t; and*
(4) *F is equivariant under the diagonal action of* $\mathrm{Out}(F_n)$ *on $X_n \times \overline{X}_n$.*

Proof. Let $\phi = M(T_0, T_1)$ be the morphism from Proposition 4.3. (Since T_1 is a minimal action, ϕ is surjective.) There is a natural path in $\overline{X}_n$ from T_0 to $T_2 = \mathscr{D}(M(T_0, T_1))$, since T_2 is in the closure of the simplex of X_n containing T_0. Applying Proposition 4.2 to ϕ, we obtain a continuous path of nontrivial semisimple actions from T_2 to T_1, given by $\alpha(t) = \mathscr{R}(\phi_{0t})$, $0 \le t \le 1$. This path will not in general be contained in the set $\mathscr{X}$ of minimal actions. However, $[l(\alpha(t))]$ is a path from T_2 to T_1 which is contained in $\mathscr{X}$ (by our identification) and, in fact, is contained in $\overline{X}_n$ [Sk, Theorem 6.7]. The composition of these two paths gives a natural path in $\overline{X}_n$ from T_0 to T_1. If $M(T_0, T_0)$ is the identity morphism, F is the trivial path at T_0. Since the morphism $M(T_0, T_1)$ is equivariant under the action of $\mathrm{Out}(F_n)$, so is the path from T_0 to T_1, and by Proposition 4.2, it varies continuously with T_0 and T_1. □

5. Finite groups of automorphisms

Theorem 5.1. *The subset X_G (resp. $\overline{X}_G$) of X_n (resp. $\overline{X}_n$) fixed by a finite subgroup G of* $\mathrm{Out}(F_n)$ *is contractible.*

Proof. By a theorem of Culler [Cu], G has a fixed point T_0 in X_n. By Theorem 4.4, we have a natural path from T_0 to any other fixed point T_1. This path is contained in X_n (resp. $\overline{X}_n$) if T_1 is. By equivariance, each such path is fixed under $\mathrm{Out}(F_n)$. Since these paths vary continuously with T_1, we can contract X_G (resp. $\overline{X}_G$) continuously along these paths to the point T_0. □

Remarks. 1. The same argument shows that the fixed point set of any finite subgroup of $\mathrm{Out}(F_n)$ in the space of all nontrivial semisimple actions of F_n on **R**-trees is contractible.

2. For a finite subgroup G of $\mathrm{Out}(F_n)$, Krstić and Vogtmann [KV] construct a subcomplex L_G of X_n on which the centralizer $C(G)$ acts with finite stabilizers and finite quotient. Their main result is the contractibility of L_G and hence an upper bound on the vcd of $C(G)$. Since L_G is an equivariant deformation retract of the fixed set X_G, Theorem 5.1 yields an alternate route to the contractibility of L_G.

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Department of Mathematics, University of California, Los Angeles, California 90024

Current address: Department of Mathematics, University of California, Riverside, California 92521

E-mail address: tadpole@ucrmath.ucr.edu