FIXED POINTS OF FINITE GROUPS OF FREE GROUP AUTOMORPHISMS

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Abstract. We construct an equivariant collection of contractions of the compactified Culler-Vogtmann outer space $\mathcal{X}_n$. As a consequence, we prove that any finite subgroup of the outer automorphism group of a free group fixes a contractible subset of $\mathcal{X}_n$.

1. Introduction

Culler and Vogtmann [CV] initiated a study of the outer automorphism group $\text{Out}(F_n)$ of the free group $F_n$ on $n$ letters by constructing a space $X_n$ upon which $\text{Out}(F_n)$ acts properly discontinuously. This space, which has since come to be known as the “outer space”, consists of projective classes of free minimal actions of $F_n$ on simplicial metric trees, where two such trees are identified if they are equivariantly isometric. The quotient of each such tree by such an action is a finite marked metric graph in which each vertex has valence $\geq 3$. Here, “metric” means that each edge has a length, and a “marking” is a preferred homotopy equivalence of the wedge $R_n$ of $n$ circles to this graph. One thus has two possible views of $X_n$; these are interchangeable. The group $\text{Out}(F_n)$ acts in the obvious way: one represents an automorphism of $F_n$ as a self-map of $R_n$ and precomposes the marking with this map.

Culler and Vogtmann showed that $X_n$ is a finite-dimensional contractible space and that the $\text{Out}(F_n)$-action has finite stabilizers and finite quotient. Furthermore, $X_n$ has a natural compactification $\overline{X}_n$ in which the boundary consists of certain actions of $F_n$ on $\mathbb{R}$-trees with cyclic edge stabilizers. It is known that $\overline{X}_n$ is contractible; this was first shown by Steiner [St] and, independently, by Skora [Sk]. Skora’s technique was to construct a path between two $\mathbb{R}$-trees, given a morphism between them. (A morphism is a map such that each segment in the domain $\mathbb{R}$-tree contains an initial arc which is mapped isometrically.) In this paper we generalize Skora’s methods to prove the following theorem:

Theorem. Let $\mathcal{X}$ be the space of nontrivial semisimple actions of $F_n$ on $\mathbb{R}$-trees. Then there is a continuous deformation $F : X_n \times \mathcal{X} \times I \to \mathcal{X}$ such that

$F(T_0, T_1, 0) = T_0$.
The result also holds if we replace $\mathcal{P}$ by the compactification $\overline{X}_n \subset \mathcal{P}$ of $X_n$. An immediate consequence is the following:

**Corollary.** The subset $X_G$ (resp. $\overline{X}_G$) of $X_n$ (resp. $\overline{X}_n$) fixed by a finite subgroup $G$ of Out$(F_n)$ is contractible.

### 2. Preliminaries

**Group actions on R-trees.** An R-tree is a metric space such that any two points are joined by a unique embedded arc, which is isometric to a finite interval in the real line. For a general exposition of the theory of group actions on R-trees, see, for example, [CM].

Let $G \times T \to T$ be an action of a finitely generated group $G$ on an R-tree. The metric on $T$ will be denoted by $d$ or $d_T$ as the context requires. The translation length function $l(g)$ determined by the action is the map $l : G \to \mathbb{R}$ defined by $l(g) = \min_{x \in T} d(x, gx)$. The characteristic set of $g$ is $T_g = \{ x \in T \mid d(x, gx) = l(g) \}$. Since $l(g) = l(xgx^{-1})$ for all $g, x \in G$, we may consider $l$ as a real-valued function on the set $C$ of conjugacy classes in $G$. Actions of $G$ on R-trees $T$ and $T'$ are projectively equivalent if there is an equivariant homeomorphism from $T$ to $T'$ such that the induced pull-back metric on $T$ differs from the original metric by a nonzero multiplicative constant.

An action is irreducible if no finite set of ends of $T$ is invariant under $G$ and dihedral if $G$ interchanges a pair of ends. The action is semisimple if it is irreducible, dihedral, or trivial.

We now describe a topology, due to Paulin [P], on the space of actions of $G$ on R-trees. An $\varepsilon$-approximation between two compact metric spaces $K$ and $L$ is a subdirect product $R \subset K \times L$ (i.e., the relation $R$ surjects onto both $K$ and $L$) such that if $x_0 R y_0$ and $x_1 R y_1$, then $|d(x_0, x_1) - d(y_0, y_1)| < \varepsilon$. Following Gromov [G], define $d_G(K, L) < \varepsilon$ if there exists an $\varepsilon$-approximation between $K$ and $L$. More generally, suppose $X$ is a G-metric space (i.e., a metric space with an associated $G$-action) and that $P$ is a finite subset of $G$. An $\varepsilon$-approximation is $P$-equivariant if whenever $g \in P$, $x \in K$, $gx \in K$, $y \in L$, and $xRy$, then $gy \in L$ and $gxRgy$.

Now fix a compact $K \subset X$, a finite subset $P$ of $G$, and $\varepsilon > 0$. Paulin [P] defines the basic neighborhood $U(X, K, P, \varepsilon)$ to be the collection of $G$-metric spaces $Y$ such that there exists a compact $L \subset Y$ and a $P$-equivariant $\varepsilon$-approximation from $K$ to $L$ and uses these as a neighborhood base for a topology on $G$-metric spaces.

Skora [Sk] generalizes this to define a topology on a space $\mathcal{E}$ of equivariant maps between $G$-metric spaces. We give the definitions here for completeness. If $\phi : X \to X'$ and $\psi : Y \to Y'$ are maps, an $\varepsilon$-approximation from $\phi$ to $\psi$ is a pair of relations $(R, R')$ such that

1. $R$ and $R'$ are $\varepsilon$-approximations from $X$ to $Y$ and from $X'$ to $Y'$ respectively; and
2. $xRy$ implies $\phi(x)R'\psi(y)$.
We endow a product $X \times X'$ with the uniform metric:

$$d((x_0, x'_0), (x_1, x'_1)) = \max\{d(x_0, x_1), d(x'_0, x'_1)\}.$$ 

Given $\varepsilon$-approximations $R$ from $X$ to $Y$ and $R'$ from $X'$ to $Y'$, one sees that $(R, R')$ is an $\varepsilon$-approximation from $X \times X'$ to $Y \times Y'$, where $(R, R')$ is the product of $R$ and $R'$ as subsets of $X \times Y$ and $X' \times Y'$ respectively.

The notion of $P$-equivariance extends naturally to the case of $\varepsilon$-approximations between maps. Skora defines a topology on a space $\mathcal{C}$ of equivariant maps between $G$-metric spaces as follows: Let $K \times K'$ be a compact subset of $X \times X'$, $P$ a finite subset of $G$, and $\varepsilon > 0$. The basic neighborhood $U(\phi, K \times K', P, \varepsilon)$ of $\phi$ is the set of all maps $\psi : Y \to Y'$ in $\mathcal{C}$ such that for some compact $L \times L' \subseteq Y \times Y'$, there is a $P$-equivariant, closed $\varepsilon$-approximation from $\phi|K$ to $\psi|L$. $\mathcal{C}$ is then given the topology generated by these basic open sets.

**A universal bundle.** Fix a group $G$ and a space $\mathcal{A}$ of actions of $G$ on $\mathbb{R}$-trees. In this section we introduce a bundle over $\mathcal{A}$ which should be thought of as an analogue of the universal Teichmüller bundle over Teichmüller space. Although it is essentially a notational convenience, this bundle will be useful in describing some constructions later and in dealing with the assorted group actions which arise.

**Definition 2.1.** Let $\mathcal{A}$ be a space of $G$-actions on $\mathbb{R}$-trees. The tree bundle $\mathcal{T}(\mathcal{A})$ over $\mathcal{A}$ is the space $\{(T, x) \mid T \in \mathcal{A} \text{ and } x \in T\}$. If the $\mathbb{R}$-trees are simplicial, we further define the vertex bundle $\mathcal{V}(\mathcal{A})$ over $\mathcal{A}$ to be the space $\{(T, v) \mid T \in \mathcal{A} \text{ and } v \text{ is a vertex of } T\}$. There is a natural $G$-action on each fiber of these bundles.

$\text{Aut}(G)$ acts naturally on $\mathcal{T}(\mathcal{A})$ by permuting the fibers. If we fix a group action $G \times T \to T$ in $\mathcal{A}$ and $\phi \in \text{Aut}(G)$, we can define the image of this action under $\phi$ by precomposing:

$$G \times T \xrightarrow{\gamma} T \xrightarrow{\gamma_0(\phi \times \text{id})} T.$$ 

In $\mathcal{T}(\mathcal{A})$, then, we define $\phi(T, x) = (T', x')$ if there exists an isometry of pointed spaces $f : (T, x) \to (T', x')$ which induces $\phi$ in the sense that the following diagram commutes:

$$
\begin{array}{ccc}
G \times T & \xrightarrow{\phi \times f} & G \times T' \\
\downarrow & & \downarrow \\
T & \xrightarrow{f} & T'
\end{array}
$$

This action restricts to an action on $\mathcal{V}(\mathcal{A})$. Note that the “vertical” action of $G$ on the fibers is simply the restriction of the $\text{Aut}(G)$ action to $\text{Inn}(G)$, and since $\text{Inn}(G)$ acts trivially on the base space, we recover the standard $\text{Out}(G)$ action on $\mathcal{A}$.

Finally, there is a natural topology on the tree bundle. $T$ and $T'$ are close if there exists an approximating relation $R$ between large compact subsets of $T$ and $T'$, such that $R$ is equivariant with respect to a large finite subset of $G$. Hence two points $(T, x)$ and $(T', x')$ in the
bundle are close if \( xR_{x'} \) for such a relation \( R \). We endow the vertex bundle with the topology it inherits from the tree bundle.

### 3. Weighted length functions

Throughout this section, \( G \) denotes a finitely generated group. All actions of \( G \) are assumed to be minimal.

We now generalize the notions of translation length function and characteristic set. Let \( G \) act on \( T \), and let \( \lambda \) be a real-valued function on \( G \). The **weighted length function** \( f_\lambda : T \to \mathbb{R} \) is defined to be \( \sup_{g \in G} \lambda(g)d(x, gx) \). We define the **length** of \( \lambda \) to be the infimum of \( f_\lambda : l(\lambda) = l_T(\lambda) = \inf_{x \in T} f_\lambda(x) \).

(We will omit the subscript if the tree is clear from the context.) The **characteristic set** of \( \lambda \) is \( \mathcal{T}_\lambda = \{ x \in T \mid f_\lambda(x) = l(\lambda) \} \).

**Remarks.** 1. For a general function \( \lambda \), it may happen that \( l(\lambda) = \infty \) and that \( \mathcal{T}_\lambda = T \). We will eventually restrict our attention to functions \( \lambda \) for which this does not occur, but remark that the definitions still make sense in this case.

2. Skora [Sk] considers the case in which \( \lambda \) is the characteristic function of a finite generating set of \( G \). The arguments here have a slightly different flavor, since the weighting functions now have infinite support.

3. The standard translation length \( l(g) \) of a group element \( g \) is recovered in this context by taking \( \lambda \) to be the function taking the value 1 at \( g \) and 0 elsewhere. Just as \( l(g) \) is invariant under conjugacy in \( G \), the length \( l(\lambda) \) is invariant under the “inner” action of \( G \) on \( \mathbb{R}^G \) by conjugation.

There is a natural action of \( \text{Aut}(G) \) on \( \mathbb{R}^G \): given \( \lambda \in \mathbb{R}^G \) and an automorphism \( \phi \) of \( G \), one can define \( \phi(\lambda) = \lambda \circ \phi^{-1} \). The following lemma is immediate.

**Lemma 3.1.** If \( \phi \in \text{Aut}(G) \), \( \lambda \in \mathbb{R}^G \), and \( f \) is an isometry from \( (T, v) \) to \( (T', v') \) inducing \( \phi \) as in the preceding section, then:

1. \( l(f(T))(\lambda) = l_T(\phi(\lambda)) \), and
2. \( f(\mathcal{T}_\lambda) = \mathcal{T}_{\phi(\lambda)} \).

In other words, the generalized notions of length function and characteristic set are equivariant. □

For the remainder of this section, we shall assume \( G \) acts freely on a simplicial \( \mathbb{R} \)-tree \( T \) (and hence \( G = F_n \)). We now choose, for each point of \( T \), a weighting function on \( G \):

**Definition.** Let \( G \times T \to T \) be a free simplicial action. For any \( x \in T \), define a weighting function \( \lambda_{T,x} : G \to \mathbb{R} \) by \( \lambda_{T,x}(g) = 1/d_T(x, gx) \) if \( g \neq \text{id} \), and \( \lambda_{T,x}(\text{id}) = 0 \).

Then, for any semisimple action \( G \times T' \to T' \), we let \( f_{T,x} : T \to \mathbb{R} \) denote the weighted length function corresponding to \( \lambda_{T,x} : f_{T,x}(y) = \sup_{g \in G} \lambda_{T,x}(g)d(y, gx) = \sup_{g \in G - \{\text{id}\}} \frac{d_T(y, gx)}{d_T(x, gx)} \).

We first analyze the behavior of the function \( f_{T,x} \) on \( T \). A priori, the supremum defining \( f_{T,x} \) may not even be finite, so this is the first issue to be dealt with.
Lemma 3.2. Suppose that $G$ acts irreducibly on $T$. Then for each $x \in T$, the function $f_{T,x} : T \to \mathbb{R}$ is:

1. finite;
2. proper; and
3. has a unique minimum at $x$.

Proof. (1) Fix $y \in T$. Then for any $g \in G$,
\[ d(y, gy) \leq d(x, y) + d(x, gx) + d(gx, gy) = d(x, gx) + 2d(x, y), \]
so that $\lambda_{T,x}(g)d(y, gy) \leq 1 + 2d(x, y)/d(x, gx)$. But the action on $T$ is free and simplicial, so $d(x, gx)$ is uniformly bounded away from zero for all $g \neq \text{id}$. Thus $\lambda_{T,x}(g)d(y, gy)$ is uniformly bounded in $g$, and thus $f_{T,x}(y) < \infty$.

(2) Let $y$ be any point in $T$ other than $x$. By the minimality of $T$, we can choose an element $g$ of $G$ such that $x$ lies on the segment joining $y$ to the axis of $G$. For such a $g$, $d(y, gy) = d(x, gx) + 2d(x, y)$. Hence $f_{T,x}(y)$ increases with $d(x, y)$. This also proves (3). □

We shall require the following lemma:

Lemma 3.3. Let $\lambda : G \to \mathbb{R}$ be a real-valued function. Suppose that $G \times T \to T$ is a free simplicial action such that the weighted length function $f_{\lambda}$ is finite and proper on $T$. Then $f_{\lambda}$ is finite and proper for any semisimple action of $G$ on an $\mathbb{R}$-tree $T'$.

Proof. Let $h : T \to T'$ be an edgewise-linear equivariant map. We use $h$ to relate the behavior of $f_{\lambda}$ on $T'$ to its behavior on $T$. Since $T$ has compact quotient, $h$ is $L$-Lipschitz for some $L > 0$. Furthermore, $h$ is onto, since the action on $T'$ is minimal. Let $z \in T$ be an $h$-preimage of $y \in T'$. Then $d_{T'}(y, gy) \leq Ld_T(z, gz)$ for any $g \in G$. Thus $f_{\lambda}(y) \leq Lf_{\lambda}(z)$. By the preceding lemma, $f_{\lambda}$ is finite on $T$, so it is finite on $T'$. That $f_{\lambda}$ is proper follows as in the preceding lemma. □

Definition 3.4. Let $\Lambda$ denote the set of real-valued functions $\lambda$ on $G$ such that the weighted length function $f_{\lambda}(x)$ is a proper, real-valued function for a free action of $G$ on a simplicial $\mathbb{R}$-tree $T$. Since any two such actions are equivariantly quasi-isometric, the preceding lemma implies that $\Lambda$ is independent of the choice of $T$. We remark that the weighted length function is automatically convex for each $\lambda \in \Lambda$, as each function $d_T(x, gx)$ is convex.

Remark. The correspondence $(T, x) \mapsto \lambda_{T,x}$ gives a continuous map from the tree bundle $\mathcal{T}(X_n)$ to $\mathbb{R}^G$. Lemma 3.2 thus says that the image of this map is contained in $\Lambda$.

Now suppose $\lambda \in \Lambda$, and let $T$ be an $\mathbb{R}$-tree equipped with a semi-simple action of $G$. Since $f_{\lambda}$ is convex and proper on $T$, its characteristic set $T_{\lambda}$ is a nonempty compact subtree of $T$. We can then choose a “base point” $b(T, \lambda)$ in $T$ by selecting the “center” of $T_{\lambda}$; that is, the unique point $y$ such that the ball at $y$ of radius $(\text{diam } T_{\lambda})/2$ is contained in $T_{\lambda}$. Hence $b(\cdot, \lambda)$ may be thought of as a section of the tree bundle $\mathcal{T}(\mathcal{X})$. The following shows that these sections are continuous and vary continuously with $\lambda$.

Proposition 3.5. Let $G = F_n$ and $\Lambda$ be as above. Let $\mathcal{X}$ denote the space of semisimple actions of $G$ on $\mathbb{R}$-trees and $\mathcal{T}(\mathcal{X})$ be the tree bundle over $\mathcal{X}$. Then the associated base point function $b : \mathcal{X} \times \Lambda \to \mathcal{T}(\mathcal{X})$ is continuous.
The proof is essentially identical to that of Proposition 5.2 in [Sk]. More generally, if $G$ is a finitely generated group, $\mathcal{H}$ a space of actions of $G$ on R-trees, and $\Lambda$ a collection of weight functions on $G$ such that $f_\lambda$ is finite and proper for each $\lambda \in \Lambda$ and each action in $\mathcal{H}$, then the same result holds.

**Lemma 3.6.** $b$ is equivariant under the action of $\text{Aut}(F_n)$ on $\mathcal{H} \times \Lambda$.

**Proof.** This follows immediately from Lemma 3.1. □

### 4. Paths between R-trees

In this section we take $G$ to be the free group $F_n$ of a fixed rank $n$. Recall [CV] that “outer space” is the space $X_n$ of projective classes of minimal free actions of $F_n$ on simplicial R-trees. A map from a simplicial R-tree to an arbitrary R-tree is transverse if it is linear on each 1-simplex.

If $f$ is any function, we will denote by $\mathcal{D}(f)$ and $\mathcal{R}(f)$ the domain and range of $f$.

**Proposition 4.1.** Let $\mathcal{H}$ be a space of nontrivial irreducible actions of $F_n$ on R-trees, and let $\mathcal{C} = \mathcal{C}(X_n, \mathcal{H})$ be the space of all equivariant transverse maps from simplicial R-trees to elements of $\mathcal{H}$. Then there is a map $B : X_n \times \mathcal{H} \to \mathcal{C}$ such that $\mathcal{D}(B(Y, T)) = Y$, $\mathcal{R}(B(Y, T)) = T$, and $B(Y, Y) = \text{Id}_{\mathcal{H}}$ for all $Y \in X_n \cap \mathcal{H}$, $T \in \mathcal{H}$. Furthermore, $B$ is equivariant with respect to the diagonal action of $\text{Out}(F_n)$ on $X_n \times \mathcal{H}$.

**Proof.** Given $Y \in X_n$ and $T \in \mathcal{H}$, we use the base point map to define the image of a vertex $v$ of $Y$: $B(Y, T)(v) = b(T, \lambda_Y, v)$. Now extend the domain of $B(Y, T)$ to all of $Y$ by mapping linearly on the edges of $Y$. Now $\lambda_Y, v$ is continuous, so $B(Y, T)$ is continuous by Lemma 3.3 and Proposition 3.5. $B(Y, T)$ is equivariant under $\text{Out}(F_n)$ by Lemma 3.6. Finally, by Lemma 3.2, the weighted length function $f_Y, v$ on $Y$ has a unique minimum at $v$, so $B(Y, Y)$ is the identity on the vertices of $Y$ and hence on all of $Y$ by linearity. □

**Remark.** In the above proposition, it is essential that the action on the source R-tree be free. If any $g \in F_n$ has a fixed point in $Y$, it is impossible in general to construct an equivariant map from $Y$ to $T$.

Recall [Sk] that an (equivariant) morphism between R-trees is an (equivariant) map such that each segment in the source R-tree has a nontrivial initial segment which is mapped isometrically onto its image. Given a surjective morphism $\phi : T_0 \to T_1$, Skora constructs morphisms $\phi_{st}$ for $0 \leq s \leq t \leq 1$ such that $\phi_{01} = \phi$, $\phi_{st} \circ \phi_{rs} = \phi_{rt}$, and $\phi_{rs} = \text{id}$ whenever $\phi = \text{id}$. We will need the following result:

**Proposition 4.2** [Sk, 4.8]. Let $G$ be a group, $\mathcal{H}$ the space of all actions of $G$ on R-trees, and $\mathcal{C}(\mathcal{H})$ the space of all morphisms between elements of $\mathcal{H}$. Then the function

$$\mathcal{C}(\mathcal{H}) \times \{(s, t) \mid 0 \leq s \leq t \leq 1\} \to \mathcal{C}(\mathcal{H})$$

defined via $(\phi, (s, t)) \mapsto \phi_{st}$ is continuous. Furthermore, this map is equivariant with respect to the natural action of $\text{Out}(F_n)$ on $\mathcal{C}(\mathcal{H})$.

Hence a morphism between two R-trees determines a canonical path between them. We now improve Proposition 4.1 to promote $B(Y, T)$ to a morphism:
Proposition 4.3. Let $\mathcal{H}$ be the space of nontrivial semisimple actions of $F_n$ on $\mathbb{R}$-trees, and let $\mathcal{C} = \mathcal{C}(X_n, \mathcal{H})$ be the space of all equivariant transverse morphisms from elements of $X_n$ to elements of $\mathcal{H}$. Then there is a map $M : X_n \times \mathcal{H} \to \mathcal{C}$ such that the following hold for all $Y \in X_n$, $T \in \mathcal{H}$:

1. $\mathcal{C}(M(Y, T)) = T$;
2. $\mathcal{C}(M(Y, T))$ is a simplicial $\mathbb{R}$-tree contained in the closure of the simplex of $X_n$ containing $Y$;
3. if $T$ is a free simplicial action, so is $\mathcal{C}(M(Y, T))$;
4. $M(Y, Y) = \text{Id}_Y$; and
5. $M$ is equivariant under the diagonal action of $\text{Out}(F_n)$.

Proof. For each 1-simplex $\sigma$ in $Y$, $B(Y, T)$ maps $\sigma$ linearly into $T$, dilating by some nonnegative number $\mu$. Multiply the metric on $\sigma$ by $\mu$. Let $Y'$ be the tree obtained from $Y$ in this way. $B(Y, T)$ then induces a morphism $M(Y, T)$ from $Y'$ to $T$. $M$ clearly varies continuously with $Y$ and $T$, and the conclusions all follow immediately from the corresponding properties of $B$. □

Theorem 4.4. There is a continuous function $F : X_n \times X_n \times I \to X_n$ such that

1. $F(T_0, T_1, 0) = T_0$; 
2. $F(T_0, T_1, 1) = T_1$; 
3. $F(T_0, T_0, t) = T_0$ for all $t$; and 
4. $F$ is equivariant under the diagonal action of $\text{Out}(F_n)$ on $X_n \times X_n$.

Proof. Let $\psi = M(T_0, T_1)$ be the morphism from Proposition 4.3. (Since $T_1$ is a minimal action, $\psi$ is surjective.) There is a natural path in $X_n$ from $T_0$ to $T_2 = \mathcal{C}(M(T_0, T_1))$, since $T_2$ is in the closure of the simplex of $X_n$ containing $T_0$. Applying Proposition 4.2 to $\psi$, we obtain a continuous path of nontrivial semisimple actions from $T_2$ to $T_1$, given by $\alpha(t) = \mathcal{C}(\phi_{\psi(t)})$, $0 \leq t \leq 1$. This path will not in general be contained in the set $\mathcal{H}$ of minimal actions. However, $[\alpha(t)]$ is a path from $T_2$ to $T_1$ which is contained in $\mathcal{H}$ (by our identification) and, in fact, is contained in $X_n$ [Sk, Theorem 6.7]. The composition of these two paths gives a natural path in $X_n$ from $T_0$ to $T_1$. If $M(T_0, T_0)$ is the identity morphism, $F$ is the trivial path at $T_0$. Since the morphism $M(T_0, T_1)$ is equivariant under the action of $\text{Out}(F_n)$, so is the path from $T_0$ to $T_1$, and by Proposition 4.2, it varies continuously with $T_0$ and $T_1$. □

5. Finite groups of automorphisms

Theorem 5.1. The subset $X_G$ (resp. $\overline{X}_G$) of $X_n$ (resp. $\overline{X}_n$) fixed by a finite subgroup $G$ of $\text{Out}(F_n)$ is contractible.

Proof. By a theorem of Culler [Cu], $G$ has a fixed point $T_0$ in $X_n$. By Theorem 4.4, we have a natural path from $T_0$ to any other fixed point $T_1$. This path is contained in $X_n$ (resp. $\overline{X}_n$) if $T_1$ is. By equivariance, each such path is fixed under $\text{Out}(F_n)$. Since these paths vary continuously with $T_1$, we can contract $X_G$ (resp. $\overline{X}_G$) continuously along these paths to the point $T_0$. □

Remarks. 1. The same argument shows that the fixed point set of any finite subgroup of $\text{Out}(F_n)$ in the space of all nontrivial semisimple actions of $F_n$ on $\mathbb{R}$-trees is contractible.
2. For a finite subgroup $G$ of $\text{Out}(F_n)$, Krstić and Vogtmann [KV] construct a subcomplex $L_G$ of $X_n$ on which the centralizer $C(G)$ acts with finite stabilizers and finite quotient. Their main result is the contractibility of $L_G$ and hence an upper bound on the vcd of $C(G)$. Since $L_G$ is an equivariant deformation retract of the fixed set $X_G$, Theorem 5.1 yields an alternate route to the contractibility of $L_G$.

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