SIMPLEXES IN RIEMANNIAN MANIFOLDS

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Abstract. Existence of a simplex with prescribed edge lengths in Euclidean, spherical, and hyperbolic spaces was studied recently. A simple sufficient condition of this existence is, roughly speaking, that the lengths do not differ too much. We extend these results to Riemannian n-manifolds \( M^n \). More precisely, we consider \( m + 1 \) points \( p_0, p_1, \ldots, p_m \) in \( M^n \), with prescribed mutual distances \( l_{ij} \) and establish a condition on the matrix \( (l_{ij}) \) under which the points \( p_i \) can be selected as freely as in \( R^n \): \( p_0 \) is a prescribed point, the shortest path \( p_0p_1 \) has a prescribed direction at \( p_0 \), the triangle \( p_0p_1p_2 \) determines a prescribed 2-dimensional direction at \( p_0 \), and so on.

1. Basic definitions and the theorem

Existence of a simplex with prescribed edge lengths in Euclidean, spherical, and hyperbolic spaces was studied in [3]. A simple sufficient condition of this existence established there is, roughly speaking, that the edge lengths do not differ too much, see [3, Theorem 2]. We deal here with \( m + 1 \) points \( p_0, p_1, \ldots, p_m \) in a Riemannian \( n \)-manifold \( M^n \), with prescribed mutual distances \( l_{ij} \) and establish a condition on the matrix \( (l_{ij}) \) under which the points \( p_i \) can be selected as freely as in \( R^n \): \( p_0 \) is a prescribed point, the shortest path \( p_0p_1 \) has a prescribed direction at \( p_0 \), the triangle \( p_0p_1p_2 \) determines a prescribed 2-dimensional direction at \( p_0 \), and so on. Our result however does not guarantee uniqueness of the points \( p_i \) (see more on that at the ends of parts A and I of §3). Note that the desired points \( p_i \) may not exist even though all the distances \( l_{ij} \) are equal and the manifold \( M^n \) is complete, noncompact, and expanding in the following sense: there exists a point \( w \in M^n \) and a constant \( c > 0 \) such that for any triangle \( awb \) with \( wa = wb \), one has \( ab > c \cdot wa \cdot \angle awb \) where \( \angle \) means angle. An appropriate example for four points in \( M^3 \) can be constructed as follows. Let \( M^2 \) be a narrow right circular cone. Its vertex \( v \) can be smoothed out later for regularity. Put \( M^3 = M^2 \times R \). One can check that \( M^3 \) is expanding if the point \( (v, 0) \) is chosen as the point \( w \). Prescribe \( l_{ij} = 1 \).
Select points \( q_0, q_1, \) and \( q_2 \in M^2 \) on a circumference centered at \( v \) such that their mutual distances in \( M^2 \) all are unit. Now put \( p_0 = (q_0, 0), p_1 = (q_1, 0), \) and \( p_2 = (q_2, 0). \) Obviously \( l_{01} = l_{02} = l_{12} = 1. \) By symmetry, the last point \( p_3 \) should be of the form \((v, h), h \in \mathbb{R}. \) Then \( l_{03} = l_{31} = l_{32} = (r^2 - h^2)^{1/2} \) where \( r \) is the radius of the above circumference. When the cone \( M^2 \) is sufficiently narrow, one has \( r > 1. \) Then \( l_{03} = l_{31} = l_{32} > 1 \) and hence the desired point \( p_3 \) does not exist when \( p_0, p_1, \) and \( p_2 \) are selected as above.

There is another subtle difference in this area between \( M^n \) and Euclidean, hyperbolic, or spherical \( n \)-space \( X_k^n \) of curvature \( k. \) Consider, say, a tetrahedron in \( X_k^n. \) If the directions of the three edges coming from its vertex are coplanar then the same is true of each other vertex of the tetrahedron. This is not so in \( M^n \) even for small tetrahedra.

By \( k \)-plane, we will mean \( X_k^n. \) The sphere \( X_1^n \) will often be denoted by \( S^n. \)

The notation \( xy \) will be used for a geodesic with ends \( x, y \) for its length and for the distance between \( x \) and \( y. \) The meaning will be specified in cases of possible confusions.

A set \( C \subset M^n \) is called convex if for each two points in \( C \) there exists a unique shortest path in \( M^n \) connecting these points and this path (which is a geodesic) belongs to \( C. \)

Let \((x_{ij})\) be a matrix with \( x_{ii} = 0, x_{ij} = x_{ji} > 0, i, j = r, r+1, \ldots, r+s. \) (We will encounter cases \( r = 0 \) and \( r = 1. \)) Such a matrix will be called allowable. Let \( q_r, q_{r+1}, \ldots, q_{r+s} \) be \( s+1 \) points in a metric space \( Y \) with the mutual distances \( q_iq_j = x_{ij}. \) The set of these \( s+1 \) points will be called a realization of \((x_{ij})\) in \( Y \) and often written down as \( q_rq_{r+1}\cdots q_{r+s}. \) Suppose \( Y = X_k^n \) with \( n \geq s. \) In case \( k > 0, \) assume also that the points \( q_i \) lie in an open semisphere of \( X_k^n. \) If their convex hull is a nondegenerate \( s \)-simplex then we say that the matrix \((x_{ij})\) and its realization are nondegenerate in \( X_k^n. \)

Let \( M^n, n \geq 2, \) be a regular Riemannian manifold and let \( e_1, e_2, \ldots, e_m, \) \( 2 \leq m \leq n, \) be pairwise orthogonal unit vectors at a point \( p \in M^n. \) The set \( \{e_1, e_2, \ldots, e_m\} \) will be called a frame at \( p. \) Suppose that an allowable matrix \((l_{ij}), \) \( i, j = 0, 1, \ldots, m, \) has a realization \( p_0p_1\cdots p_m \) in \( M^n \) such that, for each pair \( p_i, p_j, \) the manifold \( M^n \) contains a unique shortest geodesic \( p_ip_j \) of the length \( l_{ij} \) and the following conditions hold.

\[(0) \quad p_0 = p.\]
\[(1) \quad \text{The direction of the segment } p_0p_1 \text{ is } e_1.\]
\[(2) \quad \text{The direction of the segment } p_0p_2 \text{ is coplanar with } e_1 \text{ and } e_2 \text{ and forms with } e_2 \text{ an angle } < \pi/2.\]
\[(3) \quad \text{The direction of the segment } p_0p_3 \text{ is coplanar with } e_1, e_2, \text{ and } e_3 \text{ and forms with } e_3 \text{ an angle } < \pi/2.\]
\[(m) \quad \text{The direction of the segment } p_0p_m \text{ is coplanar with } e_1, e_2, \ldots, e_m \text{ and forms with } e_m \text{ an angle } < \pi/2.\]

We will say then that the realization \( p_0p_1\cdots p_m \) of \((l_{ij})\) fits the frame \( \{e_1, e_2, \ldots, e_m\} \) at \( p. \)

**Theorem.** Let \( M^n, n \geq 2, \) be a regular Riemannian \( n \)-manifold, not necessarily complete. Let \( p \in M^n, \) \( r > 0 \) be less than or equal to the convexity radius at \( p \) (see [4, §5.2] for the definition), let \( k' \) and \( k'' \) be finite lower and upper bounds of the sectional curvature in the \( r \)-neighbourhood \( N_r(p) \) of \( p, \) and let
{e_1, e_2, ..., e_m}, \ 2 \leq m \leq n, \ be \ a \ frame \ at \ p. \ Suppose \ that \ an \ allowable \ matrix \ \langle l_{ij} \rangle, \ i, \ j = 0, 1, \ldots, m, \ satisfies \ the \ following \ conditions:

(i) \ l_{0i} < r, \ i = 1, 2, \ldots, m.

(ii) For each pair of distinct i and j different from 0, there exists a non-degenerate triangle on the \( k' \)-plane \( (k'' \)-plane) with side lengths \( l_{0i}, l_{0j}, \) and \( l_{ij}. \)

(1) \ (Thus \ its \ perimeter \ is \ < 2\pi/\sqrt{k''} \ if \ k'' > 0.)

(iii) With \( \alpha'_{ij} \) \( (\alpha''_{ij}) \) \ being \ the \ angle \ of \ that \ triangle \ opposite \ to \ the \ side \ of \ the \ length \ \( l_{ij} \), each allowable matrix \( (\alpha_{ij}) \) satisfying

(2) \ \alpha'_{ij} \leq \alpha_{ij} \leq \alpha''_{ij}, \quad i, j = 1, 2, \ldots, m,

has a nondegenerate realization \( a_1a_2\cdots a_m \) in \( S^{n-1} \).

Then the matrix \( \langle l_{ij} \rangle \) has a realization \( p_0p_1\cdots p_m \) in \( M^n \) which fits the frame \{e_1, e_2, ..., e_m\}. Moreover, any realization \( p_0p_1\cdots p_k, k < m, \) of the matrix \( \langle l_{ij} \rangle \) with \( i, j = 0, 1, \ldots, k \) fitting the frame \{e_1, e_2, ..., e_k\} (we do not know if such a realization is unique) can be augmented by points \( p_{k+1}, p_{k+2}, \ldots, p_m \) such that the resulting set \( p_1p_2\cdots p_m \) is a realization of the original matrix \( \langle l_{ij} \rangle \) fitting the frame \{e_1, e_2, ..., e_m\}.

2. SOME RELATED QUESTIONS

Remark 1. Condition (ii) of the theorem is not easy to check directly. Theorem 2 in [3] yields a simple sufficient condition for (iii) to hold. Take some

(3) \ \alpha \geq \max_{i,j} \alpha''_{ij}

and suppose that

(4) \ \alpha \leq 2 \arcsin \sqrt{\frac{m}{2(m-1)}}.

Then, by [3, Theorem 2], a quantity

(5) \ \lambda = \lambda(m-1, \alpha) \in (0, \alpha)

is determined identically by the equation

(6) \ \sin \lambda = \sin \alpha [1 - f(m-1)(\cos^2 R)/\cos^2(\alpha/2)]^{1/2}

where

(7) \ f(m-1) = \begin{cases} 2/m & \text{if } m \text{ is even,} \\ 2m/(m-1)(m+1) & \text{if } m \text{ is odd} \end{cases}

and

(8) \ \sin R = \sqrt{2(m-1)/m \sin(\alpha/2)}.

(We use \( m - 1 \) as the integer argument to comply with [3].) This \( \lambda \) has the property that each allowable \( m \times m \) matrix \( (\alpha_{ij}) \) with

(9) \ \alpha_{ij} \in (\lambda, \alpha), \quad i \neq j,

is nondegenerate in \( S^{m-1} \). Thus if \( \min_{i,j} \alpha'_{ij} > \lambda = \lambda(m-1, \alpha) \) then (iii) holds.
Remark 2. A better though less convenient method to check condition (iii) arises from Theorem 1 in [3]. Put

\[ s_{ij} = \cos \alpha_{ij} - \cos \alpha_{im} \cos \alpha_{jm}, \quad i, j = 1, 2, \ldots, m - 1. \]

Theorem 1 in [3] implies that the \( m \times m \) matrix \( (\alpha_{ij}) \) of our theorem has a nondegenerate realization in \( S^{m-1} \) if and only if the \( (m - 1) \times (m - 1) \) matrix \( S = (s_{ij}) \) has a positive spectrum. To establish this property of \( S = S(\alpha_{ij}) \) for each combination of the \( m(m - 1)/2 \) arguments \( \alpha_{ij} \) in the domain \( D: \alpha_{ij}' < \alpha_{ij} < \alpha_{ij}'' \), it is enough to establish this property for one such combination and then make sure that \( \det S(\alpha_{ij}) \) stays positive in \( D \).

Remark 3. Suppose the sectional curvature is constant in \( N_r(p) \). Then one can take \( k' = k'' = k \). The matrix \( (\alpha_{ij}) \) is now unique. Its realizability can be checked with the help of Theorems 1 and 2 in [3]. Our theorem means now that each allowable matrix \( (l_{ij}) \) is freely realizable in \( N_r(p) \) with 0th vertex at \( p \) (see the next remark) if and only if it satisfies (i), (ii), and (iii). (Considering necessity of (iii), one should notice that fitting the frame \( \{e_1, e_2, \ldots, e_m\} \) implies nondegeneracy of the realization \( p_0p_1 \cdots p_m \) in \( N_r(p) \), which in turn implies nondegeneracy of the realization \( a_1 \cdots a_m \) in \( S^{n-1} \).

Let, for instance, \( M^3 = S^1 \times R^2 \). Then the convexity radius \( R(p) = \pi/2 \) for any \( p \in M^3 \). Take \( r = \pi/2 \). Consider the matrix

\[
L = \begin{pmatrix}
0 & l_{01} & l_{02} \\
l_{10} & 0 & l_{12} \\
l_{20} & l_{21} & 0
\end{pmatrix} = \begin{pmatrix}
0 & 1.58 & 1.58 \\
1.58 & 0 & 3.15 \\
1.58 & 3.15 & 0
\end{pmatrix}
\]

where \( 1.58 > \pi/2 \) and \( 3.15 > \pi \). This matrix has realizations in \( M^3 \), say, those located in \( R^2 \). The theorem, however, does not guarantee existence of any realization of \( L \) since \( l_{01} > r \) in violation of (i). Importance of (i) becomes clear if one notices that \( L \) has no realization in \( S^1 \times R^1 \subset M^3 \) which would be symmetric about \( R^1 \). At the same time, replacing 1.58 and 3.15 by 1.57 and 3.13, one gets a matrix freely realizable in \( M^3 \) (see Remark 5 for an exact definition) according to the theorem.

Remark 4. Let \( K \) be a nonempty set in \( M^n \) and let \( (l_{ij}) \) be an allowable matrix. If for any \( p \in K \) and any frame \( \{e_1, e_2, \ldots, e_m\} \), \( 2 \leq m \leq n \), at \( p \) the matrix \( (l_{ij}) \) has a realization in \( M^n \) fitting this frame, we will say that \( (l_{ij}) \) is freely realizable in \( M^n \) with 0th vertex in \( K \). The theorem gives a simple sufficient condition of such realizability. Suppose that \( \inf_{p \in K} R(p) > 0 \) where \( R(p) \) is the radius of convexity. Take a positive \( r < \inf_{p \in K} R(p) \) and let \( k', k'' \) be finite lower and upper bounds of sectional curvature in the \( r \)-neighbourhood of \( K \). Suppose that conditions (i), (ii), and (iii) hold with these \( r, k', k'' \). The theorem implies then that \( (l_{ij}) \) is freely realizable in \( M^n \) with 0th vertex in \( K \).

Remark 5. Let \( P \) be a permutation on \( \{0, 1, \ldots, m\} \). Put \( l^P_{ij} = l_{P(i)P(j)} \), \( i, j = 0, 1, \ldots, m \). If the matrix \( (l^P_{ij}) \) is freely realizable in \( M^n \) with 0th vertex in \( K \) for any \( P \), we will say that the original matrix \( (l_{ij}) \) is freely realizable in \( M^n \) with a vertex in \( K \). If \( K = M^n \) here, we will say that \( (l_{ij}) \) is freely realizable in \( M^n \). (In this case, \( M^n \) of course should be complete.)
Free realizability with 0th vertex in $K$ does not imply free realizability with a vertex in $K$. In the rest of this remark, we assume that $r$, $k'$, and $k''$ are as in the preceding remark. Note that conditions (i), (ii), and (iii) can hold for $(l_{ij})$ but fail for $(l_{ij}')$. Of course, if (i), (ii), and (iii) hold for each matrix $(l_{ij}')$ then $(l_{ij})$ is freely realizable with a vertex in $K$.

A natural question to ask in this connection is as follows. Suppose that $(l_{ij})$ is freely realizable in $M^n$ with 0th vertex in $K$. Put

$$
\tilde{l} = \max_{i} l_{0i};
$$

$$
\tilde{K} = \begin{cases} 
\{ x \in K | \rho(x, \partial K) \geq \tilde{l} \} & \text{if } \partial K \neq \emptyset, \\
M^n & \text{if } \partial K = \emptyset \text{ (then } K = M^n). 
\end{cases}
$$

Suppose $\tilde{K} \neq \emptyset$. It is guaranteed now that each realization $p_0p_1\ldots p_m$ of $(l_{ij})$ with at least one vertex, say $p_1$, in $\tilde{K}$ has $p_0 \in K$. Let $F = \{ f_1, f_2, \ldots, f_m \}$ be a frame at the point $p_1 \in \tilde{K}$. Is it possible to select $p_0 \in K$ and a frame $\{ e_1, e_2, \ldots, e_m \}$ at $p_0$ such that the realization $p_0p_1\ldots p_m$ would fit both frames? In other words, is $(l_{ij})$ freely realizable in $M^n$ with a vertex in $K$?

We do not know the answer.

Remark 6. Let $K$, $r > 0$, $k'$, and $k''$ be as in Remark 4. Let $L = (l_{ij})$, $i, j = 0, 1, \ldots, m$, be the matrix of the edge lengths of a nondegenerate Euclidean $m$-simplex. Put $L(\varepsilon) = (l_{ij}(\varepsilon))$, $\varepsilon > 0$. Then $L(\varepsilon)$ is freely realizable in $M^n$ with 0th vertex in $K$ for sufficiently small $\varepsilon$. Indeed, together with the angles $\alpha'_{ij} = \alpha'_{ij}(\varepsilon)$ and $\alpha''_{ij} = \alpha''_{ij}(\varepsilon)$ on $k'$- and $k''$-planes for the matrix $L(\varepsilon)$, consider also the appropriate angles $\alpha_{ij}^0$ on Euclidean plane. (They do not depend on $\varepsilon$.)

The matrix $(\alpha_{ij}^0)$ is nondegenerate in $S^{m-1}$ since $L$ is nondegenerate in $R^m$. Obviously, $\alpha'_{ij} \rightarrow \alpha_{ij}^0$, $\alpha''_{ij} \rightarrow \alpha_{ij}^0$ as $\varepsilon \rightarrow 0$. Then, for sufficiently small $\varepsilon$, any matrix $(\alpha_{ij})$ with $\alpha_{ij} \in [\alpha_{ij}(\varepsilon), \alpha''_{ij}(\varepsilon)]$ is arbitrarily close to the nondegenerate $(\alpha_{ij}^0)$. Since nondegenerate matrices form an open set in the appropriate matrix space (see [3, Corollary of Theorem 1]), these matrices $(\alpha_{ij})$ are nondegenerate. Now the theorem implies that $L(\varepsilon)$ is freely realizable in $M^n$ with 0th vertex in $K$.

Applying this observation to each permuted matrix $l_{ij}'$ (see Remark 5), one will see also that $L(\varepsilon)$ is freely realizable in $M^n$ with a vertex in $K$ when $\varepsilon$ is sufficiently small.

Remark 7. Note finally that the theorem does not assume triangle inequalities involving $l_{ij}$, $l_{ik}$, $l_{jk}$ in which the index 0 does not appear among $i$, $j$, $k$. Those triangle inequalities follow from the theorem, i.e., from realizability of the matrix $(l_{ij})$.

3. Proof of the theorem

A. One may consider only the case $k = m - 1$ since points can be added one at a time. We use induction by $m$. If $m = 2$, one obviously can select $p_0$ and $p_1$ fitting the frame $\{ e_1 \}$. (This selection is unique in this particular case.) Let $p(\phi)$ be such that the length $p_0p(\phi) = l_{02}$, and the direction of the segment $p_0p(\phi)$ is coplanar with $e_1$ and $e_2$ and forms an angle $\phi \in [0, \pi]$ with $e_1$ and an angle $\leq \pi/2$ with $e_2$. The distance $p_1p(\phi)$ changes continuously with $\phi$. 
from $p_1 p(0) = \|l_{01} - l_{02}\| \leq l_{12}$ to $p_1 p(\pi) = l_{01} + l_{02} \geq l_{12}$. Then $p_1 p(\phi^*) = l_{12}$ for some $\phi^* \in [0, \pi]$. If $\phi^* = \pi$ then $l_{12} = l_{01} + l_{02}$, which is impossible for the nondegenerate triangles mentioned in (ii). Thus $\phi^* \neq \pi$. If $\phi^* = 0$, then either $l_{01} - l_{02} = l_{12}$ or $l_{02} - l_{01} = l_{12}$, which is also impossible due to (ii). Hence $\phi^* \in (0, \pi)$ and the point $p_2 = p(\phi^*)$ is a desirable one. (We do not know if $\phi^*$ and $p_2$ are unique.) Thus the theorem, including the statement on augmentation, holds for $m = 2$.

B. Suppose now that the theorem holds for $m - 1 \geq 2$ in place of $m$. Along with the matrix $L = (l_{ij}), \ i, j = 0, 1, \ldots, m$, we will consider three other matrices: $L_m$ with is obtained from $L$ by deleting its $m$th, i.e., the last, row and column; $L_{m-1}$ obtained from $L$ by deleting its $(m-1)$st row and column; and $L_{m-1} \ell$ obtained from $L$ by deleting its last two rows and last two columns. Similarly, we introduce three modifications, $A_m$, $A_{m-1}$, and $A_{m-1} \ell$, of a matrix $A = (a_{ij}), \ i, j = 1, 2, \ldots, m$. Note that since $A$ has a nondegenerate realization in $S^{n-1}$ for any choice of its elements $a_{ij} \in [a_{i}'j, a_{i}'j']$, the same is true of $A_m$, $A_{m-1}$, and $A_{m-1} \ell$. By our induction assumption, there exists a realization $p_0 p_1 \cdots p_m$ of $L_m$ fitting the frame $\{e_1, e_2, \ldots, e_{m-1}\}$. For $\phi \in [0, \pi]$, denote by $e(\phi)$ the unit vector coplanar to $e_{m-1}$ and $e_m$ forming an angle $\phi$ with $e_{m-1}$ and an angle $\leq \pi/2$ with $e_m$. Obviously the part $p_0 p_1 \cdots p_m$ of the last realization is a realization of $L_{m-1} \ell$. By our induction assumption, this realization $p_0 p_1 \cdots p_m \ell$ can be augmented by a point $p(\phi)$ such that the resulting set $p_0 p_1 \cdots p_m \ell p(\phi)$ is a realization of $L_{m-1}$ fitting the frame $\{e_1, e_2, \ldots, e_{m-2}, e(\phi)\}$. Denote by $a_1, \ldots, a_{m-1}, a(\phi)$ the directions of the segments $p_0 p_1, \ldots, p_0 p_{m-1}, p_0 p(\phi)$ at $p_0$. We now specify the entries $a_{ij}$ of the matrix $A$ above as follows. We assume $a_{ij}$ to be the distance $a_i a_j$ on the sphere $S^{n-1}$ of directions at $p_0$ for $i, j \leq m - 1$, i.e.,

$$a_{ij} = a_i a_j = \angle p_i p_0 p_j, \quad i, j = 1, 2, \ldots, m - 1.$$  

We put

$$a_{im} = a_{mi} = a_i a(\phi), \quad i = 1, 2, \ldots, m - 1, \quad a_{mm} = 0.$$  

Thus $a_1 \cdots a_{m-1} a(\phi)$ is now a realization of the matrix $A$ in $S^{n-1}$ while $a_1 \cdots a_{m-1}, a_1 \cdots a_{m-2} a(\phi)$, and $a_1 \cdots a_{m-2}$ are realizations of $A_m$, $A_{m-1}$, and $A_{m-1} \ell$. Note that in case $M^n = X_k^n$, the entries $a_{im}$ do not depend on $\phi$ except for $a_{m-1} \ell$.

C. We make now an important reference to comparison theorems for triangles by Alexandrow and Toponogov. That will be the only substantial reference to Riemannian Geometry in this paper. Since $l_{0i} < r$, all our points $p_0, p_1, \ldots, p_{m-1}$, $p(\phi)$ and the segments between them lie in $N_r(p_0)$. According to [2, §6.4.2, Theorem and Remark 3], Toponogov's Theorem can be stated as follows.

**Theorem (V. A. Toponogov).** Let $C$ be a convex set in $M^n$, $n \geq 2$, and $k'$ be a lower bound of the sectional curvature at points of $C$. Then for any triangle made of shortest paths in $C$ there exists a triangle in the $k'$-plane with the same side lengths such that the angles $\alpha, \beta, \gamma$ of the triangle in $C$ and the corresponding angles $\alpha', \beta', \gamma'$ of the triangle in the $k'$-plane satisfy

$$\alpha' \leq \alpha, \quad \beta' \leq \beta, \quad \gamma' \leq \gamma.$$
Since $N_r(p_0)$ is convex, the theorem applies to it and yields

(13) $\alpha'_{ij} \leq \alpha_{ij}, \quad i, j = 1, 2, \ldots, m - 1,$
(14) $\alpha'_{im} \leq \alpha_{im} = a_ia(\phi) = \angle p_ip_0p(\phi)$ for $i \leq m - 2$ and $\phi \in [0, \pi].$

The local comparisons of the angles of triangles like those in (13) and (14) were actually understood prior to Toponogov's global results, e.g., by Alexandrow.

Note that $\alpha_{m-1m} = \angle p_{m-1}p_0p(\phi)$ is not involved in either (13) or (14) since, generally speaking, $p_{m-1}p(\phi) \neq l_{m-1m}.$ (We are just working towards the appropriate equality.) Denote by $B$ the closed metric ball centered at $p_0$ whose radius is $\max_{1 \leq i \leq m} l_{i0}.$ Since this radius is $< r,$ the ball $B$ is convex. According to [1, §1.7b]), $B$ is a domain of type $R_{k''}$ (defined in [1, §1.4]). It follows from [1, the end of §1.6 and §1.4c)] that, for any triangle in $B$ of perimeter $< 2\pi\sqrt{k''},$ if $k'' > 0,$ the triangle in the $k''$-plane with the same side lengths satisfies

(15) $\alpha < \alpha'', \quad \beta < \beta'', \quad \gamma < \gamma''$

where $\alpha, \beta, \gamma$ are the angles of the triangle in $B$ and $\alpha'', \beta'', \gamma''$ are the corresponding angles of the triangle on the $k''$-plane. Due to (1), the estimate (15) can be applied to the triangles $p_ip_0p_j$ and $p_ip_0p(\phi)$ resulting in

(16) $\alpha'_{ij} \leq \alpha''_{ij}, \quad i, j = 1, 2, \ldots, m - 1,$
(17) $\alpha'_{im} = \angle p_ip_0p(\phi) \leq \alpha''_{im}$ for $i \leq m - 2$ and $\phi \in [0, \pi].$

The relations (13), (14), (16), and (17) mean that the off-diagonal elements of the matrices $A_m$ and $A_{m-1}$ satisfy condition (2). Therefore their realizations $a_1 \cdots a_{m-1}$ and $a_1 \cdots a_{m-2}a(\phi)$ are nondegenerate.

D. It will be convenient to associate with these two realizations the nondegenerate spherical $(m-2)$-simplexes $a_1a_2\cdots a_{m-1}$ and $a_1a_2\cdots a_{m-2}a(\phi)$ of which the first one is immovable while the second one varies with $\phi.$ The variation, however, is not rotation about the common $(m-3)$-face $a_1a_2\cdots a_{m-2}$ since the lengths of the edges $a_ia(\phi), \ i \leq m - 2,$ generally speaking depend on $\phi$ (see (11)) unless $M^n = X^k.$ In subsections E, F, G, and H we are going to watch the distance $a(\phi) = a_{m-1}a(\phi)$ in $S^n-1$ as $\phi$ varies on $[0, \pi].$

E. Let $S^{m-2} \subset S^{n-1}$ be the sphere determined by $a_1a_2\cdots a_{m-1}.$ Denote by $H_0$ and $H_n$ the two closed hemispheres of $S^{m-2}$ whose common boundary is the sphere $S^{m-3}$ determined by $a_1a_2\cdots a_{m-2};$ see Figure 1 on the next page. Since the simplex $a_1a_2\cdots a_{m-1}$ is nondegenerate, the point $a_{m-1} \notin S^{m-3}.$ One may assume that

(18) $a_{m-1} \in \text{relint } H_0 = H_0 \setminus S^{m-3}.$

Put

(19) $F_0 = \{x \in H_0|xa_i \in [\alpha'_{im}, \alpha''_{im}], \ i = 1, 2, \ldots, m-2\};$
(20) $F_n = \{x \in H_n|xa_i \in [\alpha'_{im}, \alpha''_{im}], \ i = 1, 2, \ldots, m-2\}.$

Let us show that

(21) $a(0) \in F_0, \quad a(\pi) \in F_n.$
Figure 1

Since \( p_0p_1 \cdots p_{m-1} \) and \( p_0p_1 \cdots p_{m-2}p(0) \) both fit the same frame \( \{e_1, e_2, \ldots, e_{m-1} = e(0)\} \), the direction \( a(0) \) of the segment \( p_0p(0) \) lies in \( H_0 \) as the point \( a_{m-1} \) does according to (18). The distances \( a_i a(0) = \alpha_{i,m}(0), \ i \leq m - 2 \) (see (11)), satisfy (2) according to (14) and (17). Thus \( a(0) \in F_0 \). Similarly \( a(\pi) \in F_\pi \).

F. Denote by \( G \) the spherical shell

\[
G = \{x \in S^{n-1}|\alpha'_{m-1,m} \leq xa_{m-1} \leq \alpha''_{m-1,m}\}.
\]

Let us show that

\[
F_0 \cap G = \emptyset; \quad F_\pi \cap G = \emptyset.
\]

Suppose to the contrary that, say, \( F_0 \cap G \ni z \). Then the mutual distances of the \( m \) points \( a_1, \ldots, a_{m-1}, z \) in \( S^{n-1} \) satisfy (2), and hence \( a_1 \cdots a_{m-1}z \) should be a nondegenerate \((m - 1)\)-simplex. On the other hand, these \( m \) points lie in \( S^{m-2} \).

G. We prove now that the spherical shell

\[
G \text{ separates } F_0 \text{ and } F_\pi.
\]

Suppose to the contrary that, for instance,

\[
F_0 \cup F_\pi \subset B \overset{\text{def}}{=} \{x \in S^{n-1}|xa_{m-1} < \alpha'_{m-1,m}\}.
\]

Consider the matrix

\[
A' \overset{\text{def}}{=} \begin{pmatrix}
0 & \alpha_{12} & \ldots & \alpha_{1,m-1} & \alpha'_{1,m} \\
\alpha_{21} & 0 & \ldots & \alpha_{2,m-1} & \alpha'_{2,m} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_{m-1,1} & \alpha_{m-1,2} & \ldots & 0 & \alpha'_{m-1,m} \\
\alpha'_{m,1} & \alpha'_{m,2} & \ldots & \alpha'_{m,m-1} & 0
\end{pmatrix}
\]
where $\alpha_{ij}$ are defined by (10). Due to (13) and (16), condition (2) holds for $A'$. Then $A'$ has a nondegenerate realization. One obviously may assume that it lies in the $(m - 1)$-sphere $S^{m-1} \subset S^{n-1}$ determined by the points $e_1, \ldots, e_m$ and that the first $m - 1$ points of the realization are $a_1, a_2, \ldots, a_{m-1}$. Denote by $b$ the last point of this realization. Then

(27) \hspace{1cm} b \in \partial B .

As the point $b$ rotates in $S^{m-1}$ about $S^{m-3}$ (determined by $a_1 \cdots a_{m-2}$), it travels a circumference $C$ (nondegenerate since $b \notin S^{m-3}$), see Figure 1. The 2-sphere $S^2$ determined by $C$ is orthogonal to $S^{m-3}$ and thus to $S^{m-2}$. Therefore the great circle $S^1 = S^2 \cap S^{m-2}$ includes a diameter $c_0c_\pi$ of the circle $K = S^2 \cap B$, see Figure 1. (By a diameter, we mean here a longest geodesic in $K$ that can be longer than $\pi$ when the radius $\alpha'_{m-1,m}$ of $B$ is $> \pi/2$.) Due to (27),

(28) \hspace{1cm} b \in C \cap \partial K .

Note that the points $b_0$ and $b_\pi$ which make up $C \cap S^1 = C \cap S^{m-2}$ satisfy

(29) \hspace{1cm} a_i b_0 = a_i b_\pi = \alpha'_i m , \quad i = 1, 2, \ldots, m-2 .

Thus each of them is either in $F_0$ or $F_\pi$; however, if one is in $F_0$ then the other one should be in $F_\pi$ since $b_0$ and $b_\pi$ are distinct and symmetric about $S^{m-3}$. One may assume that

(30) \hspace{1cm} b_0 \in F_0 ; \quad b_\pi \in F_\pi .

By contrary assumption (25), $b_0$ and $b_\pi$ lie in the open ball $B$. Therefore $b_0$ and $b_\pi$ are interior points of the diameter $c_0c_\pi$, see Figure 1. Obviously the circumference $C$ is orthogonal to $c_0c_\pi$ at $b_0$ and $b_\pi$. Then $C$ and $\partial K$ cannot intersect contrary to (28). The case $F_0 \cup F_\pi \subset S^{m-1}\setminus(B \cup G)$ reduces to a contradiction, similarly, which proves (24).

H. Since the points $b_0, b_\pi \in S^{m-2}$ are symmetric about $S^{m-3}$ and (see (30)) $b_0 \in F_0$ thus lying on the same side of $S^{m-3}$ with $a_{m-1}$, the distance

(31) \hspace{1cm} b_0 a_{m-1} \leq b_\pi a_{m-1} .

Now if $F_\pi \subset B$ and, by (24), $F_0 \subset S^{m-1}\setminus(G \cup B)$ then by (30) the distance $b_\pi a_{m-1} < b_0 a_{m-1}$ contrary to (31). Hence

(32) \hspace{1cm} F_0 \subset B , \quad F_\pi \subset S^{m-1}\setminus(G \cup B) .

Now come back to the distance $\alpha(\phi) = a_{m-1} a(\phi)$ singled out in subsection D. Due to (32) and (21), one has

(33) \hspace{1cm} \alpha(0) < \alpha'_{m-1,m} \leq \alpha''_{m-1,m} < \alpha(\pi) .

I. Now we watch the distance $p_{m-1}p(\phi)$ in $M^n$. For the triangle $p_{m-1}p_0p(\phi)$, consider in the $k'$-plane ($k''$-plane) a triangle with the same side lengths. Denote by $\alpha'(\phi)$ ($\alpha''(\phi)$) its angle opposite to the side of the length $p_{m-1}p(\phi)$. Due to (12) and (15),

(34) \hspace{1cm} \alpha'(\phi) \leq \alpha(\phi) \leq \alpha''(\phi) .

Suppose now that $p_{m-1}p(0) \geq l_{m-1}m$. Due to (34) and geometry of the $k$-plane, one has then $\alpha(0) \geq \alpha'(0) \geq \alpha'_{m-1,m}$ contrary to (33). Thus $p_{m-1}p(0) < l_{m-1}m$. 

Similarly, \( p_{m-1}p(\pi) > l_{m-1}m \). By continuity, \( p_{m-1}p(\phi^*) = l_{m-1}m \) at some \( \phi^* \in (0, \pi) \).

Thus an arbitrary realization \( p_0p_1\cdots p_{m-1} \) of \( L_m \) fitting the frame \( \{e_1, e_2, \ldots, e_{m-1}\} \) has been augmented by the point \( p_m = p(\phi^*) \) such that \( p_0p_1\cdots p_m \) is a realization of \( L \). Obviously this realization fits the frame \( \{e_1, e_2, \ldots, e_m\} \). This completes the proof. (Again, we do not know if \( \phi^* \) and \( p_m \) are unique.)

**REFERENCES**


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