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A CENTRAL LIMIT THEOREM ON HEISENBERG TYPE GROUPS. II

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ABSTRACT. We present a Liapounov type central limit theorem for random variables associated to a commutative Banach algebra of "radial" measures on Heisenberg type groups. This theorem improves on a result presented by the author in Proc. Amer. Math. Soc. **113** (1991), 529–536.

0. INTRODUCTION

In [Oh] we introduced a commutative algebra of "radial", bounded, Borel measures on Heisenberg type groups (H-type groups). For probability measures in this algebra satisfying certain integrability conditions we proved a central limit theorem analogous to one of the classical Euclidean versions [Oh, Theorem 4.1]). The proof exploited explicit formulas for the Gelfand transform on the above mentioned commutative algebra.

In this paper we present a Liapounov type version of the central limit theorem on H-type groups. The awkward integrability hypothesis of [Oh] is replaced by a more standard third moment integrability hypothesis.

As in [Oh] we exploit some of the many parallels between analysis on H-type groups and Euclidean analysis (cf. [Fa]). In particular use is made of homogeneous Taylor polynomials on H-type groups.

The main result is presented in §2. Section 1 is devoted to preliminaries on H-type groups, homogeneous Taylor polynomials, group valued random variables, and the heat semigroup that provides us with a notion of normal distributions.

1. PRELIMINARIES

A group of type H is a connected, simply connected, real Lie group whose Lie algebra is of type H. A Lie algebra $\mathfrak{n}$ is of type H if $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$; $\mathfrak{v}, \mathfrak{z}$ real Euclidean spaces, with a Lie algebra structure such that $\mathfrak{z}$ is the center of $\mathfrak{n}$ and for all $v \in \mathfrak{v}$ of length one, ad_v is a surjective isometry of the orthogonal complement of $\ker ad_v$ in $\mathfrak{v}$, onto $\mathfrak{z}$.

Let $\mathcal{N}$ be a type H group and $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ its Lie algebra. There is a natural dilation structure on $\mathcal{N}$. For $s > 0$ define $\delta_s(v, z) = (sv, s^2 z)$.

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We fix a basis $X_1, X_2, \dots, X_n$ for $\mathfrak{n}$ consisting of eigenvectors for the dilations δ_s with eigenvalues $r^{d_1}, \dots, r^{d_n}$ ($d_i = 1$ or 2) in such a way that $X_1, X_2, \dots, X_{\dim \mathfrak{v}}$ forms a basis for $\mathfrak{v}$. For a multi-index $I = (i_1, i_2, \dots, i_n)$ let $d(I) = d_1 i_1 + d_2 i_2 + \cdots + d_n i_n$. $d(I)$ is the homogeneous degree of $X^I = X_1^{i_1} \cdots X_n^{i_n}$.

The left Taylor polynomial of f at g of homogeneous degree a is the unique homogeneous polynomial P of homogeneous degree less than or equal to a such that $X^I P(0) = X^I f(g)$ for all multi-indices I with $d(I) \le a$.

In [FS, Theorem 1.42] Folland and Stein prove that if $f \in C^{k+1}(\mathcal{N})$ with bounded derivatives of order $(k+1)$ and P_g is the left Taylor polynomial of homogeneous degree k, then

$$|f(gg') - P_g(g')| \le K|g|^{k+1} \quad \text{for } g, g' \in \mathcal{N}.$$

(Here $|g|$ denotes the homogeneous norm of g on $\mathcal{N}$.)

For example, on the three-dimensional Heisenberg whose Lie algebra is spanned by three vectors X, Y, Z, $[X, Y] = Z$, the left Taylor polynomial of f at g of homogeneous degree 2 is given by

$$\begin{aligned} P_g(x, y, z) &= f(g) + (Xf)(g)x + (Yf)(g)y + (Zf)(g)z \\ &\quad + \frac{(X^2 f)(g)}{2!}x^2 + \frac{(Y^2 f)(g)}{2!}y^2 + \frac{((XY + YX)f)(g)}{2!}xy \end{aligned}$$

where $g \in \mathcal{N}$. Thus it follows that

$$|f(gg') - P_g(g')| \le K|g'|^3 \quad \text{for } g, g' \in \mathcal{N},$$

provided f has uniformly bounded third derivatives.

An $\mathcal{N}$-valued random variable is a measurable function from some probability space $(\Omega, \mathcal{F}, \mathcal{P})$ to $\mathcal{N}$. For each $\mathcal{N}$ random variable ξ we can define a probability measure μ_ξ on $\mathcal{N}$ by $\mu_\xi(A) = \mathcal{P}(\xi^{-1}(A))$, $A \subset \mathcal{N}$.

If $\varphi: \mathcal{N} \to \mathbf{R}$ we define the φ expectation of the random variable ξ to be

$$\varepsilon_\varphi(\xi) = \int_{\mathcal{N}} \varphi(g)\, d\mu_\xi(g).$$

For $F: \mathcal{N} \to \mathcal{N}$ we can define a random variable $F(\xi)$ by composition. The φ expectation of this random variable is given by

$$\varepsilon_\varphi(F(\xi)) = \int_{\mathcal{N}} \varphi(F(g))\, d\mu_\xi(g).$$

If φ is one of the coordinate functions, $\varphi(x_1, x_2, \dots, x_n) = x_i$ for some $0 \le i \le n$, then ε_{x_i} will denote the corresponding expectation.

A measurable function $\alpha: \Omega \times \Omega \to \mathcal{N} \times \mathcal{N}$ is a vector valued $\mathcal{N}$-random variable. Let μ_α^2 be the corresponding probability measure on $\mathcal{N} \times \mathcal{N}$. The prime example of this that we will use is $\alpha(\omega_1, \omega_2) = (\xi(\omega_1), \eta(\omega_2))$ for two independent random variables ξ and η, in which case μ_α^2 equals the product of the measures μ_ξ and μ_η.

Let $F: \mathcal{N} \times \mathcal{N} \to \mathcal{N}$. Then

$$\varepsilon_\varphi(F(\alpha)) = \iint_{\mathcal{N} \times \mathcal{N}} \varphi(F(g, g'))\, d\mu_\xi\, d\mu_\eta.$$

For $F(g, g') = g \cdot g'$ we have

$$\varepsilon_\varphi(F(\alpha)) = \iint_{\mathcal{N}\times\mathcal{N}} \varphi(g \cdot g')\, d\mu_\xi\, d\mu_\eta.$$

This generalizes in a straightforward manner to products of more than two independent random variables. (For more on group-valued random variables see [He].)

A bounded measure $\mu \in M_b(\mathcal{N})$ is said to be $\mathfrak{v}$ radial if $d\mu(Av, z) = d\mu(v, z)$ for all $A \in O(\mathfrak{v})$, the orthogonal group of $\mathfrak{v}$. Let $M_b(\mathcal{N})^\sharp$ denote the Banach algebra generated by the $\mathfrak{v}$-radial measures. In [Oh] we showed that $M_b(\mathcal{N})^\sharp$ is a commutative Banach algebra.

Let $\triangle$ be the usual Laplacian on $\mathfrak{v}$. We will denote by $\{p_t\}_{t>0}$ the semigroup of solutions of the heat equation corresponding to $\triangle$ on the group $\mathcal{N}$. The following properties of p_t are well known (cf. [Hu]):

(1) $p_t > 0$.
(2) p_t is $\mathfrak{v}$-radial and in $C^\infty(\mathcal{N})$.
(3) $\iint p_t(v, z)\, dv\, dz = 1$.

If $t \cdot (v, z)$ denotes dilation of (v, z) by t then it is well known that the heat semigroup satisfies

$$p_t(v, z) = t^{-Q/2} p_1(t^{-1/2} \cdot (v, z)),$$

where $Q = \dim \mathfrak{v} + 2 \dim \mathfrak{z}$ is the homogeneous dimension of the group. Furthermore it is well known that the heat semigroup is rapidly decaying at infinity. It follows that

(1.1)
$$\begin{aligned}\int_{\mathcal{N}} |(v, z)|^3 p_t(v, z)\, dv dz &= \int_{\mathcal{N}} |(v, z)|^3 t^{-Q/2} p_1(t^{-1/2} \cdot (v, z))\, dv dz \\ &= \int_{\mathcal{N}} |t^{1/2} \cdot (v, z)|^3 p_1(v, z)\, dv dz = t^{3/2} \int_{\mathcal{N}} |(v, z)|^3 p_1(v, z)\, dv dz.\end{aligned}$$

A similar calculation shows that

$$\int_{\mathcal{N}} |v|^2 p_t(v, z)\, dv dz = t \int_{\mathcal{N}} |v|^2 p_1(v, z)\, dv dz. \tag{1.2}$$

Equations (1.1) and (1.2) imply that

$$\int_{\mathcal{N}} |(v, z)|^3 p_t((v, z))\, dv dz = c \left(\int_{\mathcal{N}} |v|^2 p_t(v, z)\, dv dz \right)^{3/2} \tag{1.3}$$

for some constant c. We will exploit this relationship in the sequel.

2. Main result

In this section we present our central limit theorem. The statement and proof are based on Liapounov's and Lindeberg's theorem and proof, respectively (cf. [Ch]). We follow the notation used in [Ch] whenever possible. In the sequel expressions of the form ξ / s, ξ an $\mathcal{N}$-valued random variable and s a positive real number should be interpreted as the $\mathcal{N}$-valued random variable given by composition of ξ with $\delta_{s^{-1}}$.

Theorem 2.1. *Let $\{\xi_j\}$ be a sequence of independent $\mathcal{N}$-valued random variables such that*

(1) $\mu_{\xi_j} \in M_b(\mathcal{N})^\sharp$;
(2) $\varepsilon_\varphi(\xi_j) = 0$ *for all* φ *of the form* $\varphi(v, z) = \langle z, z' \rangle$, $z' \in \mathfrak{z}$;
(3) $\sigma^2(\xi_j) = \sigma_j^2 = \varepsilon_\varphi(\xi_j) < \infty$, *where* $\varphi(v, z) = |v|^2$;
(4) $\varepsilon(|\xi_j|^3) = \gamma_j = \varepsilon_\varphi(\xi_j) < \infty$, *where* $\varphi(v, z) = |(v, z)|^3$.

Set

$$S_m = \prod_{j=1}^m \xi_j, \qquad s_m^2 = \sum_{j=1}^m \sigma_j^2, \qquad \Gamma_m = \sum_{j=1}^m \gamma_j.$$

If $\Gamma_m / s_m^3 \longrightarrow 0$ *as* $m \longrightarrow \infty$ *then* $\mu_{S_m/s_m} \longrightarrow p_1\, dv dz$ *weakly, where* p_1 *is the element of the heat semigroup corresponding to* $t = 1$.

Proof. The idea is to approximate $\xi_1 \cdot \xi_2 \cdots \xi_m$ by replacing one ξ at a time with a comparable "normal" random variable ζ as follows: Let $\{\zeta_j\}_{j=1}^\infty$ be $\mathcal{N}$-valued random variables having absolutely continuous distributions with Radon-Nikodym derivatives p_{σ_j} (from heat semigroup). Let all the ξ_j's and ζ_j's be totally independent. Set

$$\eta_j = \zeta_1 \cdots \zeta_{j-1} \cdot \xi_{j+1} \cdots \xi_m, \qquad 1 \le j \le m,$$

with the convention that

$$\eta_1 = \xi_2 \cdots \xi_m, \qquad \eta_m = \zeta_1 \cdots \zeta_{m-1}.$$

Let $f : \mathcal{N} \to \mathcal{N}$ be C^3 with bounded derivatives of orders up to and including three. Since all the measures μ_{ξ_j}, μ_{η_j}, and μ_{ζ_j} commute, we have

$$\begin{aligned} &\varepsilon_{x_i}\left\{ f\left(\frac{\xi_1 \cdots \xi_m}{s_m}\right)\right\} - \varepsilon_{x_i}\left\{ f\left(\frac{\zeta_1 \cdots \zeta_m}{s_m}\right)\right\} \\ &\quad = \sum_{j=1}^m \left[\varepsilon_{x_i}\left\{ f\left(\frac{\xi_j \eta_j}{s_m}\right)\right\} - \varepsilon_{x_i}\left\{ f\left(\frac{\zeta_j \eta_j}{s_m}\right)\right\}\right], \end{aligned} \tag{2.1}$$

for all $1 \le i \le n$.

We would like to estimate the terms in the right-hand side of (2.1). Let $f^1, \dots, f^n$ be the components of f and $P_g^1, \dots, P_g^n$ the corresponding homogeneous Taylor polynomials of degree 2 at g. It follows from the definition of the expectation of an $\mathcal{N}$-valued random variable and the Taylor polynomial that

$$\begin{aligned} |\varepsilon_{x_i}\{f(\xi\eta)\} - \varepsilon_{x_i}\{P_\eta(\xi)\}| &= \left| \iint_{\mathcal{N}\times\mathcal{N}} (f^i(gg') - P_g^i(g'))\, d\mu_\xi\, d\mu_\eta \right| \\ &\le \iint_{\mathcal{N}\times\mathcal{N}} |f^i(gg') - P_g^i(g')|\, d\mu_\xi\, d\mu_\eta \le M \int_{\mathcal{N}} |g'|^3\, d\mu_\xi = M\varepsilon\{|\xi|^3\} \end{aligned}$$

where M represents a constant that depends on f and i. A similar argument can be carried out with ζ replacing ξ. Putting this all together we obtain

$$|\varepsilon_{x_i}\{f(\xi\eta)\} - \varepsilon_{x_i}\{f(\zeta\eta)\} + \varepsilon_{x_i}\{P_\eta(\zeta)\} - \varepsilon_{x_i}\{P_\eta(\xi)\}| \le M\varepsilon\{|\xi|^3 + |\zeta|^3\}. \tag{2.2}$$

Our choice of ζ_j implies $\varepsilon_{x_i}\{P_{\eta_j}(\zeta_j)\} = \varepsilon_{x_i}\{P_{\eta_j}(\xi_j)\}$: To see this we note that

$$P_g^i(g') = f^i(g) + \sum_{k=1}^n (X_k f^i)(g) x_k + \frac{1}{2} \sum_{k,l} (X_k X_l f^i)(g) x_j x_k,$$

where $g' = (x_1, x_2, \dots, x_n)$, and where the last sum is over $k, l = 1, 2, \dots, \dim \mathfrak{v}$. It follows that

$$\begin{aligned}\varepsilon_{x_i}\{P_{\eta_j}(\xi_j)\} &= \iint_{\mathcal{N}\times\mathcal{N}} P_g^i(g')\,d\mu_{\xi_j}(g')\,d\mu_{\eta_j}(g)\\ &= \int_{\mathcal{N}} f^i(g)\,d\mu_{\eta_j}(g) + \sum_{k=1}^{n}\left(\int_{\mathcal{N}}(X_k f^i)(g)\,d\mu_{\eta_j}(g)\cdot\int_{\mathcal{N}} x_k\,d\mu_{\xi_j}(g')\right)\\ &\quad + \frac{1}{2}\sum_{k,l}\left(\int_{\mathcal{N}}(X_k X_l f^i)(g)\,d\mu_{\eta_j}(g)\cdot\int_{\mathcal{N}} x_k x_l\,d\mu_{\xi_j}(g')\right).\end{aligned}$$

The terms in the first sum are equal to zero as a consequence of hypotheses (1) and (2) of Theorem 2.1. Terms in the second sum are equal to zero when $k \neq l$: $x_k x_l$ is integrable with respect to μ_ξ as a consequence of hypothesis (3) in Theorem 2.1 and the simple observation that $|x_k x_l| \le \frac{1}{2}(x_k^2 + x_l^2)$. Since μ_ξ is $\mathfrak{v}$-radial, a rotation of π radians in the $x_k x_l$-plane yields

$$\int_{\mathcal{N}} x_k x_l\,d\mu_{\xi_j}(g') = -\int_{\mathcal{N}} x_l x_k\,d\mu_{\xi_j}(g').$$

Hence

$$\varepsilon_{x_i}\{P_{\eta_j}(\xi_j)\} = \int_{\mathcal{N}} f^i(g)\,d\mu_{\eta_j}(g) + \frac{1}{2}\dim\mathfrak{v}\cdot\sigma_j^2\cdot\sum_{k=1}^{\dim\mathfrak{v}}\int_{\mathcal{N}}(X_k^2 f^i)(g)\,d\mu_{\eta_j}(g).$$

Similar considerations lead to the the same value for $\varepsilon_{x_i}\{P_{\eta_j}(\zeta_j)\}$.

Using this, the inequality in (2.2) becomes

$$|\varepsilon_{x_i}\{f(\xi\eta)\} - \varepsilon_{x_i}\{f(\zeta\eta)\}| \le M\varepsilon\{|\xi|^3 + |\zeta|^3\}.$$

Substituting ξ_j/s_m, η_j/s_m, ζ_j/s_m for ξ, η, ζ into this inequality and returning to (2.1), we obtain

$$\begin{aligned}&\left|\varepsilon_{x_i}\left\{f\left(\frac{\xi_1\cdots\xi_m}{s_m}\right)\right\} - \varepsilon_{x_i}\left\{f\left(\frac{\zeta_1\cdots\zeta_m}{s_m}\right)\right\}\right|\\ &\quad\le M\sum_{j=1}^{m}\left\{\frac{\varepsilon\{|\xi_j|^3\}}{s_m^3} + \frac{\varepsilon\{|\zeta_j|^3\}}{s_m^3}\right\} = M\sum_{j=1}^{m}\left\{\frac{\gamma_j}{s_m^3} + \frac{\varepsilon\{|\zeta_j|^3\}}{s_m^3}\right\}\\ &\quad\le M'\sum_{j=1}^{m}\frac{\gamma_j}{s_m^3} = M'\frac{\Gamma_m}{s_m^3}.\end{aligned}$$

In this last inequality we have used the fact that $\sigma_j^3 \le \gamma_j$ (Hölder's inequality with $p = 3/2$) and that $\varepsilon\{|\zeta_j|^3\} = c\sigma_j^3$ for some constant c (follows directly from the relationship in (1.3) and the definitions of these expectations).

Thus we have shown that for all $f : \mathcal{N} \to \mathcal{N}$ with bounded derivatives of orders up to and including three, and for all $i = 1, 2, \dots, n$,

$$\left|\varepsilon_{x_i}\left\{f\left(\frac{S_m}{s_m}\right)\right\} - \varepsilon_{x_i}\{f(N)\}\right| \le O\left(\frac{\Gamma_m}{s_m^3}\right),$$

where N denotes a random variable with probability distribution p_1. These functions are dense $C_0(\mathcal{N})$. The last inequality is equivalent to

$$\left|\int_{\mathcal{N}} f^i(g)\,d(\mu_{\xi_1/s_m} * \mu_{\xi_2/s_m} * \cdots * \mu_{\xi_m/s_m}) - \int_{\mathcal{N}} f^i(g)p_1(g)\,dg\right| \le O\left(\frac{\Gamma_m}{s_m^3}\right)$$

since by the definition of convolution

$$\int_{\mathcal{N}\cdots\mathcal{N}}\cdots\int f^i(g_1\cdots g_m)\,d\mu_{\xi_1/s_m}\cdots d\mu_{\xi_n/s_m}=\int_{\mathcal{N}} f^i(g)\,d(\mu_{\xi_1/s_m}*\cdots*\mu_{\xi_m/s_m}).$$

Thus we obtain the required weak convergence. □

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