A CENTRAL LIMIT THEOREM
ON HEISENBERG TYPE GROUPS. II

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(Communicated by Palle E. T. Jorgensen)

Abstract. We present a Liapounov type central limit theorem for random
variables associated to a commutative Banach algebra of "radial" measures on
Heisenberg type groups. This theorem improves on a result presented by the

0. Introduction

In [Oh] we introduced a commutative algebra of "radial", bounded, Borel
measures on Heisenberg type groups (H-type groups). For probability measures
in this algebra satisfying certain integrability conditions we proved a central
limit theorem analogous to one of the classical Euclidean versions [Oh, Theorem
4.1]). The proof exploited explicit formulas for the Gelfand transform on the
above mentioned commutative algebra.

In this paper we present a Liapounov type version of the central limit theorem
on H-type groups. The awkward integrability hypothesis of [Oh] is replaced by
a more standard third moment integrability hypothesis.

As in [Oh] we exploit some of the many parallels between analysis on H-
type groups and Euclidean analysis (cf. [Fa]). In particular use is made of
homogeneous Taylor polynomials on H-type groups.

The main result is presented in §2. Section 1 is devoted to preliminaries on H-
type groups, homogeneous Taylor polynomials, group valued random variables,
and the heat semigroup that provides us with a notion of normal distributions.

1. Preliminaries

A group of type H is a connected, simply connected, real Lie group whose
Lie algebra is of type H. A Lie algebra η is of type H if η = v ⊕ 3; v, 3 real
Euclidean spaces, with a Lie algebra structure such that 3 is the center of η
and for all v ∈ v of length one, ad_v is a surjective isometry of the orthogonal
complement of ker ad_v in v, onto 3.

Let N be a type H group and η = v ⊕ 3 its Lie algebra. There is a natural
dilation structure on N. For s > 0 define δ_s(v, z) = (sv, s^2 z).

Received by the editors August 20, 1991 and, in revised form, November 30, 1991.
1991 Mathematics Subject Classification. 22E27, 60B15; Secondary 43A80.
We fix a basis $X_1, X_2, \ldots, X_n$ for $n$ consisting of eigenvectors for the dilations $\delta_v$ with eigenvalues $r_1, \ldots, r_n$ ($d_i = 1$ or 2) in such a way that $X_1, X_2, \ldots, X_{d_1}$ forms a basis for $v$. For a multi-index $I = (i_1, i_2, \ldots, i_n)$ let $d(I) = d_1 i_1 + d_2 i_2 + \cdots + d_n i_n$. $d(I)$ is the homogeneous degree of $X^I = X_1^{i_1} \cdots X_n^{i_n}$.

The left Taylor polynomial of $f$ at $g$ of homogeneous degree $a$ is the unique homogeneous polynomial $P$ of homogeneous degree less than or equal to $a$ such that $X^l P(0) = X^l f(g)$ for all multi-indices $l$ with $d(l) \leq a$.

In [FS, Theorem 1.42] Folland and Stein prove that if $f \in C^{k+1}(\mathcal{N})$ with bounded derivatives of order $(k + 1)$ and $P_g$ is the left Taylor polynomial of homogeneous degree $k$, then

$$|f(g, g') - P_g(g')| \leq K |g|^{k+1} \text{ for } g, g' \in \mathcal{N}.$$  

(Here $|g|$ denotes the homogeneous norm of $g$ on $\mathcal{N}$.)

For example, on the three-dimensional Heisenberg whose Lie algebra is spanned by three vectors $X, Y, Z$, $[X, Y] = Z$, the left Taylor polynomial of $f$ at $g$ of homogeneous degree 2 is given by

$$P_g(x, y, z) = f(g) + (Xf)(g)x + (Yf)(g)y + (Zf)(g)z + \frac{(X^2f)(g)}{2!} x^2 + \frac{(Y^2f)(g)}{2!} y^2 + \frac{((XY + YX)f)(g)}{2!} xy$$

where $g \in \mathcal{N}$. Thus it follows that

$$|f(g, g') - P_g(g')| \leq K |g'|^3 \text{ for } g, g' \in \mathcal{N},$$

provided $f$ has uniformly bounded third derivatives.

An $\mathcal{N}$-valued random variable is a measurable function from some probability space $(\Omega, \mathcal{T}, \mathcal{P})$ to $\mathcal{N}$. For each $\mathcal{N}$ random variable $\xi$ we can define a probability measure $\mu_\xi$ on $\mathcal{N}$ by $\mu_\xi(A) = \mathcal{P}(\xi^{-1}(A))$, $A \subset \mathcal{N}$.

If $\varphi : \mathcal{N} \rightarrow \mathbb{R}$ we define the $\varphi$ expectation of the random variable $\xi$ to be

$$e_\varphi(\xi) = \int_\mathcal{N} \varphi(g) d\mu_\xi(g).$$

For $F : \mathcal{N} \rightarrow \mathcal{N}$ we can define a random variable $F(\xi)$ by composition. The $\varphi$ expectation of this random variable is given by

$$e_\varphi(F(\xi)) = \int_\mathcal{N} \varphi(F(g)) d\mu_\xi(g).$$

If $\varphi$ is one of the coordinate functions, $\varphi(x_1, x_2, \ldots, x_n) = x_i$ for some $0 \leq i \leq n$, then $e_{x_i}$ will denote the corresponding expectation.

A measurable function $\alpha : \Omega \times \Omega \rightarrow \mathcal{N} \times \mathcal{N}$ is a vector valued $\mathcal{N}$-random variable. Let $\mu_{\alpha, \xi}$ be the corresponding probability measure on $\mathcal{N} \times \mathcal{N}$. The prime example of this that we will use is $\alpha(\omega_1, \omega_2) = (\xi(\omega_1), \eta(\omega_2))$ for two independent random variables $\xi$ and $\eta$, in which case $\mu_{\alpha, \xi}$ equals the product of the measures $\mu_\xi$ and $\mu_\eta$.

Let $F : \mathcal{N} \times \mathcal{N} \rightarrow \mathcal{N}$. Then

$$e_\varphi(F(\alpha)) = \int_{\mathcal{N} \times \mathcal{N}} \varphi(F(g, g')) d\mu_\xi d\mu_\eta.$$
For $F(g, g') = g \cdot g'$ we have
\[ e \phi(F(\alpha)) = \int_{\mathcal{N} \times \mathcal{N}} \phi(g \cdot g') \, d\mu_{\xi} \, d\mu_{\eta}. \]

This generalizes in a straightforward manner to products of more than two independent random variables. (For more on group-valued random variables see [He].)

A bounded measure $\mu \in M_b(\mathcal{N})$ is said to be $v$ radial if $d\mu(Av, z) = d\mu(v, z)$ for all $A \in O(v)$, the orthogonal group of $v$. Let $M_b(\mathcal{N})^v$ denote the Banach algebra generated by the $v$-radial measures. In [Oh] we showed that $M_b(\mathcal{N})^v$ is a commutative Banach algebra.

Let $\Delta$ be the usual Laplacian on $v$. We will denote by $\{p_t\}_{t>0}$ the semigroup of solutions of the heat equation corresponding to $\Delta$ on the group $\mathcal{N}$. The following properties of $p_t$ are well known (cf. [Hu]):

1. $p_t > 0$.
2. $p_t$ is $v$-radial and in $C^\infty(\mathcal{N})$.
3. $\int_{\mathcal{N}} p_t(v, z) \, dv \, dz = 1$.

If $t \cdot (v, z)$ denotes dilation of $(v, z)$ by $t$ then it is well known that the heat semigroup satisfies
\[ p_t(v, z) = t^{-Q/2} p_1(t^{-1/2} \cdot (v, z)), \]
where $Q = \dim v + 2 \dim z$ is the homogeneous dimension of the group. Furthermore it is well known that the heat semigroup is rapidly decaying at infinity. It follows that
\[ \int_{\mathcal{N}} |(v, z)|^3 p_t(v, z) \, dv \, dz = \int_{\mathcal{N}} |(v, z)|^3 t^{-Q/2} p_1(t^{-1/2} \cdot (v, z)) \, dv \, dz \]
\[ = \int_{\mathcal{N}} |t^{1/2} \cdot (v, z)|^3 p_1(v, z) \, dv \, dz = t^{3/2} \int_{\mathcal{N}} |(v, z)|^3 p_1(v, z) \, dv \, dz. \]

A similar calculation shows that
\[ \int_{\mathcal{N}} |v|^2 p_t(v, z) \, dv \, dz = t \int_{\mathcal{N}} |v|^2 p_1(v, z) \, dv \, dz. \]

Equations (1.1) and (1.2) imply that
\[ \int_{\mathcal{N}} |(v, z)|^3 p_t((v, z)) \, dv \, dz = c \left( \int_{\mathcal{N}} |v|^2 p_t(v, z) \, dv \, dz \right)^{3/2} \]
for some constant $c$. We will exploit this relationship in the sequel.

2. Main result

In this section we present our central limit theorem. The statement and proof are based on Liapounov's and Lindeberg's theorem and proof, respectively (cf. [Ch]). We follow the notation used in [Ch] whenever possible. In the sequel expressions of the form $\xi / s$, $\xi$ an $\mathcal{N}$-valued random variable and $s$ a positive real number should be interpreted as the $\mathcal{N}$-valued random variable given by composition of $\xi$ with $\delta_{s^{-1}}$. 
Theorem 2.1. Let \( \{\xi_j\} \) be a sequence of independent \( \mathcal{N} \)-valued random variables such that

1. \( \mu_{\xi_j} \in \mathcal{M}_b(\mathcal{N})^4; \)
2. \( \varepsilon_{\phi}(\xi_j) = 0 \) for all \( \phi \) of the form \( \phi(v, z) = (z, z') \), \( z' \in \mathcal{N}; \)
3. \( \sigma^2(\xi_j) = \sigma_j^2 = \varepsilon_{\phi}(\xi_j) < \infty \), where \( \phi(v, z) = \|v\|^2; \)
4. \( \varepsilon(|\phi|^3) = \gamma_j = \varepsilon_{\phi}(\xi_j) < \infty \), where \( \phi(v, z) = \|v \times z\|^3. \)

Set

\[
S_m = \prod_{j=1}^{m} \xi_j, \quad s_j^2 = \sum_{j=1}^{m} \sigma_j^2, \quad \Gamma_m = \sum_{j=1}^{m} \gamma_j.
\]

If \( \Gamma_m/s_m^3 \to 0 \) as \( m \to \infty \) then \( \mu_{S_m/s_m} \to p_1 \) weakly, where \( p_1 \) is the element of the heat semigroup corresponding to \( t = 1 \).

Proof. The idea is to approximate \( \xi_1 \cdot \xi_2 \cdots \xi_m \) by replacing one \( \xi_j \) at a time with a comparable "normal" random variable \( \xi \) as follows: Let \( \{\xi_j\}_{j=1}^\infty \) be \( \mathcal{N} \)-valued random variables having absolutely continuous distributions with Radon-Nikodym derivatives \( \rho_{\xi_j} \) (from heat semigroup). Let all the \( \xi_j \)'s and \( \xi_j \)'s be totally independent. Set

\[
\eta_j = \xi_1 \cdots \xi_{j-1} \cdot \xi_{j+1} \cdots \xi_m, \quad 1 \leq j \leq m,
\]

with the convention that

\[
\eta_1 = \xi_2 \cdots \xi_m, \quad \eta_m = \xi_1 \cdots \xi_{m-1}.
\]

Let \( f : \mathcal{N} \to \mathcal{N} \) be \( C^3 \) with bounded derivatives of orders up to and including three. Since all the measures \( \mu_{\xi_j}, \mu_{\eta_j}, \text{ and } \mu_{\xi_j} \) commute, we have

\[
\varepsilon_{\eta_i} \left\{ f\left( \frac{\xi_1 \cdots \xi_m}{s_m} \right) \right\} - \varepsilon_{\eta_i} \left\{ f\left( \frac{\xi_1 \cdots \xi_m}{s_m} \right) \right\} = \sum_{j=1}^{m} \left[ \varepsilon_{\eta_i} \left\{ f\left( \frac{\xi_1 \cdots \xi_j \eta_j}{s_m} \right) \right\} - \varepsilon_{\eta_i} \left\{ f\left( \frac{\xi_1 \cdots \xi_j \eta_j}{s_m} \right) \right\} \right],
\]

for all \( 1 \leq i \leq n \).

We would like to estimate the terms in the right-hand side of (2.1). Let \( f^i, \ldots, f^n \) be the components of \( f \) and \( P_{g_i}^1, \ldots, P_{g_i}^n \) the corresponding homogeneous Taylor polynomials of degree 2 at \( g \). It follows from the definition of the expectation of an \( \mathcal{N} \)-valued random variable and the Taylor polynomial that

\[
\left| \varepsilon_{\eta_i} \{ f(\xi) \} - \varepsilon_{\eta_i} \{ P_{g}^i(\xi) \} \right| = \left| \int_{\mathcal{N} \times \mathcal{N}} (f^i(gg') - P_{g_i}^i(g')) d\mu_{\eta} d\mu_{\xi} \right|
\]

\[
\leq \int_{\mathcal{N} \times \mathcal{N}} |f^i(gg') - P_{g_i}^i(g')| d\mu_{\eta} d\mu_{\xi} \leq M \int_{\mathcal{N}} |g'|^3 d\mu_{\xi} = M \varepsilon \{ |\xi|^3 \}
\]

where \( M \) represents a constant that depends on \( f \) and \( i \). A similar argument can be carried out with \( \xi \) replacing \( \xi \). Putting this all together we obtain

\[
|\varepsilon_{\eta_i} \{ f(\xi) \} - \varepsilon_{\eta_i} \{ f(\xi) \} + \varepsilon_{\eta_i} \{ P_{g}^i(\xi) \} - \varepsilon_{\eta_i} \{ P_{g}^i(\xi) \} | \leq M \varepsilon \{ |\xi|^3 + |\xi|^3 \}.\]
where \( g' = (x_1, x_2, \ldots, x_n) \), and where the last sum is over \( k, l = 1, 2, \ldots, \dim \nu \). It follows that

\[
\varepsilon_{x_i} \{ P_{\eta_j}(\xi_j) \} = \int_{\mathcal{N}} f^i(g') d\mu_{\eta_j}(g') d\mu_{\eta_j}(g)
\]

\[
= \int_{\mathcal{N}} f^i(g') d\mu_{\eta_j}(g) + \sum_{k=1}^{n} \left( \int_{\mathcal{N}} (X_k f^i)(g) d\mu_{\eta_j}(g) \cdot \int_{\mathcal{N}} x_k d\mu_{\xi_j}(g') \right)
\]

\[+ \frac{1}{2} \sum_{k,l} \left( \int_{\mathcal{N}} (X_k X_l f^i)(g) d\mu_{\eta_j}(g) \cdot \int_{\mathcal{N}} x_k x_l d\mu_{\xi_j}(g') \right).
\]

The terms in the first sum are equal to zero as a consequence of hypotheses (1) and (2) of Theorem 2.1. Terms in the second sum are equal to zero when \( k \neq l \): \( x_k x_l \) is integrable with respect to \( \mu_{\xi_j} \) as a consequence of hypothesis (3) in Theorem 2.1 and the simple observation that \(|x_k x_l| \leq \frac{1}{2}(x_k^2 + x_l^2)\). Since \( \mu_{\xi_j} \) is \( \nu \)-radial, a rotation of \( \pi \) radians in the \( x_k x_l \)-plane yields

\[
\int_{\mathcal{N}} x_k x_l d\mu_{\xi_j}(g') = - \int_{\mathcal{N}} x_l x_k d\mu_{\xi_j}(g').
\]

Hence

\[
\varepsilon_{x_i} \{ P_{\eta_j}(\xi_j) \} = \int_{\mathcal{N}} f^i(g') d\mu_{\eta_j}(g) + \frac{1}{2} \dim \nu \cdot \sigma^2 \cdot \sum_{k=1}^{\dim \nu} \int_{\mathcal{N}} (X_k^2 f^i)(g) d\mu_{\eta_j}(g).
\]

Similar considerations lead to the the same value for \( \varepsilon_{x_i} \{ P_{\eta_j}(\xi_j) \} \).

Using this, the inequality in (2.2) becomes

\[
|\varepsilon_{x_i} \{ f(\xi_j) \} - \varepsilon_{x_i} \{ f(\eta_j) \}| \leq M \varepsilon \{ |\xi_j|^3 + |\eta_j|^3 \}.
\]

Substituting \( \xi_j / s_m, \eta_j / s_m, \zeta_j / s_m \) for \( \xi, \eta, \zeta \) into this inequality and returning to (2.1), we obtain

\[
\left| \varepsilon_{x_i} \left\{ f \left( \frac{\xi \cdots \xi_m}{s_m} \right) \right\} - \varepsilon_{x_i} \left\{ f \left( \frac{\xi \cdots \xi_m}{s_m} \right) \right\} \right|
\]

\[
\leq M \sum_{j=1}^{m} \left\{ \varepsilon \{ |\xi_j|^3 \} \right\} + \varepsilon \{ |\xi_j|^3 \} = M \sum_{j=1}^{m} \left\{ \frac{\gamma_j}{s_m^3} + \frac{\varepsilon \{ |\xi_j|^3 \}}{s_m^3} \right\}
\]

\[
\leq M' \sum_{j=1}^{m} \frac{\gamma_j}{s_m^3} = M' \frac{\Gamma_m}{s_m^3}.
\]

In this last inequality we have used the fact that \( \sigma_j^3 \leq \gamma_j \) (Hölder's inequality with \( p = 3/2 \)) and that \( \varepsilon \{ |\xi_j|^3 \} = c \sigma_j^3 \) for some constant \( c \) (follows directly from the relationship in (1.3) and the definitions of these expectations).

Thus we have shown that for all \( f : \mathcal{N} \to \mathcal{N} \) with bounded derivatives of orders up to and including three, and for all \( i = 1, 2, \ldots, n \),

\[
\left| \varepsilon_{x_i} \left\{ f \left( \frac{S_m}{s_m} \right) \right\} - \varepsilon_{x_i} \{ f(N) \} \right| \leq O \left( \frac{\Gamma_m}{s_m^3} \right),
\]

where \( N \) denotes a random variable with probability distribution \( p_1 \). These functions are dense \( C_0(\mathcal{N}) \). The last inequality is equivalent to

\[
\left| \int_{\mathcal{N}} f^i(g) d(\mu_{\xi_1/s_m} \ast \mu_{\xi_2/s_m} \ast \cdots \ast \mu_{\xi_m/s_m}) - \int_{\mathcal{N}} f^i(g)p_1(g) dg \right| \leq O \left( \frac{\Gamma_m}{s_m^3} \right)
\]

where \( g' = (x_1, x_2, \ldots, x_n) \), and where the last sum is over \( k, l = 1, 2, \ldots, \dim \nu \). It follows that

\[
\varepsilon_{x_i} \{ P_{\eta_j}(\xi_j) \} = \int_{\mathcal{N}} f^i(g') d\mu_{\eta_j}(g') d\mu_{\eta_j}(g)
\]

\[
= \int_{\mathcal{N}} f^i(g') d\mu_{\eta_j}(g) + \sum_{k=1}^{n} \left( \int_{\mathcal{N}} (X_k f^i)(g) d\mu_{\eta_j}(g) \cdot \int_{\mathcal{N}} x_k d\mu_{\xi_j}(g') \right)
\]

\[+ \frac{1}{2} \sum_{k,l} \left( \int_{\mathcal{N}} (X_k X_l f^i)(g) d\mu_{\eta_j}(g) \cdot \int_{\mathcal{N}} x_k x_l d\mu_{\xi_j}(g') \right).
\]
since by the definition of convolution
\[\int \cdots \int f^i(g_1 \cdots g_m) d\mu_{\frac{\xi_1}{s_m}} \cdots d\mu_{\frac{\xi_m}{s_m}} = \int f^i(g) d(\mu_{\frac{\xi_1}{s_m}} \ast \cdots \ast \mu_{\frac{\xi_m}{s_m}}).\]

Thus we obtain the required weak convergence. □

We would like to thank Michael Lacey for suggesting this approach to the central limit theorem.

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