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COMPLETE PURE INJECTIVITY AND ENDOMORPHISM RINGS

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ABSTRACT. It is shown that if M is a finitely presented completely pure injective object in a locally finitely generated Grothendieck category $\mathbf{C}$ such that $S = \mathrm{End}_{\mathbf{C}} M$ is von Neumann regular, then S is semisimple. This is a generalized version of a well-known theorem of Osofsky, which includes also a result of Damiano on PCI-rings. As an application, we obtain a characterization of right hereditary rings with finitely presented injective hull.

In [11, 12] Osofsky showed that a ring, all of whose cyclic right modules are injective, is semisimple (Artinian). Faith [6] studied the structure of right PCI-rings, i.e., rings whose proper right cyclic modules are injective, and he left open the question of whether right PCI-rings must be right Noetherian. In [3] Damiano gave an affirmative answer to Faith's question. The key result in [3] was the fact that a proper cyclic finitely presented module M_R over a right PCI-domain R has a semisimple endomorphism ring S [3, Proposition]. Damiano's proof uses the von Neumann regularity of S that was observed earlier by Faith [6] and a modification of a constructive technique of Osofsky [12].

In this note we prove a general version for Grothendieck categories of Osofsky's theorem [11, 12], which includes also the above-mentioned result of Damiano. Furthermore, our arguments provide a simple proof of this result. We show that if M is a finitely presented completely (pure) M-injective object in a locally finitely generated Grothendieck category $\mathbf{C}$ such that $S = \mathrm{End}_{\mathbf{C}} M$ is von Neumann regular, then S is semisimple. Consequently, if M is a projective completely injective object in $\mathbf{C}$, then $M = \bigoplus_{i\in I} A_i$, where $\mathrm{End}_{\mathbf{C}} A_i$ are division rings and the subobjects of A_i are linearly ordered. As an application to rings, we obtain a characterization of the right hereditary rings R such that the injective hull $E(R_R)$ is finitely presented. This extends a result of Colby and Rutter [2]. Note that even in the module case, Damiano's arguments cannot be applied for proving our main theorem.

Let $\mathbf{C}$ be a Grothendieck category. An object M of $\mathbf{C}$ is finitely presented if it is finitely generated and every epimorphism $A \to M$, where A is finitely

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generated, has a finitely generated kernel. $\mathbf{C}$ is said to be locally finitely generated if it has a family of finitely generated generators. A short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathbf{C}$ will be called a pure sequence when the induced morphism p: $\mathrm{Hom}_{\mathbf{C}}(F, Y) \to \mathrm{Hom}_{\mathbf{C}}(F, Z)$ is an epimorphism for every finitely presented object F of $\mathbf{C}$. In this case, X is called a pure subobject of Y. An object E of $\mathbf{C}$ is called pure injective when it has the injectivity property with respect to all pure sequences in $\mathbf{C}$ (cf. [16]). If E and M are objects of $\mathbf{C}$ and if E is injective with respect to all pure sequences with middle term M, then we will say that E is pure M-injective. An object M is called completely injective (resp. completely pure M-injective) provided every quotient of M is injective (resp. pure M-injective).

Clearly, an object E is pure injective in $\mathbf{C}$ iff E is pure M-injective for every object M in $\mathbf{C}$. However, in some sense pure M-injective objects are far from pure injectivity. If M is pure injective in $\mathbf{C}$ and $S = \mathrm{End}_{\mathbf{C}} M$, then it is well known that $S/J(S)$ is von Neumann regular (see, e.g., [16, Corollary 1.6]). Suppose that R is a right noetherian ring. Then it is easy to see that a right ideal A of R is a pure submodule of R_R iff A is a direct summand of R_R. Thus every right R-module (in particular, R_R) is pure R-injective. But indeed $R/J(R)$ need not be a von Neumann regular ring.

Now we are ready to prove our main result.

Theorem 1. *Let $\mathbf{C}$ be a locally finitely generated Grothendieck category and M a finitely presented object of $\mathbf{C}$ which is completely pure M-injective and has a von Neumann regular endomorphism ring S. Then S is a semisimple ring.*

Proof. Using Osofsky's theorem [11, 12], it will be enough to prove that each cyclic right S-module is injective. By [9, Theorem VI.3.1], the functor $\mathrm{Hom}_{\mathbf{C}}(M, -)$ from $\mathbf{C}$ to Mod-S has a left adjoint, which we denote by $- \otimes_S M$: Mod-S $\to \mathbf{C}$, and it is not difficult to check that $S \otimes_S M \cong M$ canonically. Let $f: K \to C$ be an S-homomorphism from a right ideal K of S to a cyclic right S-module C. We must show that f has an extension to a homomorphism $g: S \to C$. We may write $K = \varinjlim K_i$, where $\{K_i\}_I$ is the direct system of all the finitely generated right ideals of S contained in K, and, similarly, $C = \varinjlim C_j$, where the C_j's are finitely presented cyclic right S-modules. Since S is von Neumann regular, the K_i's and the C_j's are direct summands of S_S. Thus, since $- \otimes_S M$ is an additive functor, we have that for each $i \in I$ and $j \in J$, $K_i \otimes_S M$ and $C_j \otimes_S M$ are isomorphic to direct summands of M. Clearly, the adjunction morphisms

$$K_i \to \mathrm{Hom}_{\mathbf{C}}(M, K_i \otimes_S M), \qquad C_j \to \mathrm{Hom}_{\mathbf{C}}(M, C_j \otimes_S M)$$

are isomorphisms. The functor $- \otimes_S M$, being a left adjoint, preserves colimits and, in particular, direct limits. Also, since M is a finitely presented object of $\mathbf{C}$, the functor $\mathrm{Hom}_{\mathbf{C}}(M, -)$ preserves direct limits (see [17, Proposition V.3.4]). Thus we see that the canonical homomorphisms

$$K \to \mathrm{Hom}_{\mathbf{C}}(M, K \otimes_S M), \qquad C \to \mathrm{Hom}_{\mathbf{C}}(M, C \otimes_S M)$$

are isomorphisms, and from this it follows that f can be identified with the homomorphism $\mathrm{Hom}_{\mathbf{C}}(M, f \otimes_S M)$.

On the other hand, if we take an epimorphism $S \to C \to 0$ in Mod-S, we see that since $- \otimes_S M$ is right exact, $C \otimes_S M$ is a quotient object of M in $\mathbf{C}$

and hence is pure M-injective. The induced morphism $K \otimes_S M \to S \otimes_S M \cong M$ is the direct limit of the split monomorphisms $K_i \otimes_S M \to S \otimes_S M$ and hence is a pure monomophism (see, e.g., [19, 33.8]). Thus we have in $\mathbf{C}$ a diagram with exact row

$$\begin{array}{ccc} 0 \to K \otimes_S M & \to & S \otimes_S M \cong M \\ {\scriptstyle f \otimes_S M} \downarrow & \swarrow {\scriptstyle h} & \\ C \otimes_S M & & \end{array}$$

which can be completed by the pure M-injectivity of $C \otimes_S M$. By applying the functor $\mathrm{Hom}_{\mathbf{C}}(M, -)$ to this diagram, we obtain the S-homomorphism $\mathrm{Hom}_{\mathbf{C}}(M, h) \colon S \to C$ which extends $f \colon K \to C$. This shows that C is indeed an injective right S-module.

Remarks. (a) It is easy to see that, in the above proof, $C \otimes_S M$ is a quotient of M by a pure submodule. Thus Theorem 1 remains true if we merely assume that every quotient of M by a pure subobject is pure M-injective. There is a good supply of modules satisfying this condition; for example, all the pure injective modules over a ring of right pure global dimension ≤ 1 (and, in particular, over any countable ring [7, Théorème 7.10]). Observe also that, in fact, the same arguments show that for a finitely presented object M satisfying the above condition, every flat cyclic right S-module is pure S-injective (without assuming that S is regular).

(b) It is natural to ask whether the finitely presented condition of M in Theorem 1 could be weakened. In particular, it would be interesting to know if Theorem 1 still remains valid when M is finitely generated. We remark that, in fact, the above arguments will work if M is assumed to be finitely generated and the functor $\mathrm{Hom}_{\mathbf{C}}(M, -)$ preserves direct limits of objects which are isomorphic to direct summands of M.

Corollary 2. *Let $\mathbf{C}$ be a locally finitely generated Grothendieck category and M a finitely generated projective completely injective object in $\mathbf{C}$. Then $S = \mathrm{End}_{\mathbf{C}} M$ is semisimple.*

Proof. Clearly M is finitely presented. Let f be any element in S. Then $\mathrm{Im}(f)$ is injective and hence is a direct summand of M. It follows that $\mathrm{Im}(f)$ is projective, so $\mathrm{Ker}(f)$ is a direct summand of M. Thus S is a von Neumann regular ring (see, e.g., [17, exercise 10, p. 110]). By Theorem 1, S is semisimple.

The next result is well known for modules over a ring (see, e.g., [8, Corollary 13.6.7]).

Lemma 3. *Let M be a projective injective object in a locally finitely generated Grothendieck category $\mathbf{C}$. Then M is a coproduct of finitely generated objects.*

Proof. A categorical version of Kaplansky's theorem on projective modules was proved in [13, Lemma 3.8]. Using this, the module-theoretic arguments of [8, Corollary 13.6.7] may be carried over verbatim.

Corollary 4. *Let $\mathbf{C}$ be a locally finitely generated Grothendieck category and M a projective completely injective object in $\mathbf{C}$. Then $M = \bigoplus_{i \in I} A_i$, where for each $i \in I$, $S_i = \mathrm{End}_{\mathbf{C}} A_i$ is a division ring and the subobjects of A_i are linearly ordered.*

Proof. By Lemma 3, $M = \bigoplus_{j\in J} M_j$, where each M_j is finitely generated. By Corollary 2, $\mathrm{End}_{\mathbf{C}} M_j$ is semisimple, hence M_j is a finite direct sum of indecomposable subobjects (see, e.g., [17, Proposition XIV.1.7]). Therefore, we have $M = \bigoplus_{i\in I} A_i$, where each A_i is an indecomposable object. Clearly $S_i = \mathrm{End}_{\mathbf{C}} A_i$ is a division ring, which implies that A_i has a unique maximal subobject K_i containing every proper subobject of A_i (e.g., [19, 19.7]). Now a standard argument, similar to the proof of [14, Proposition 3], shows that the subobjects of A_i are linearly ordered.

Let R be an associative ring with identity and M_R a unitary right R-module. Denote by $\sigma[M]$ the full subcategory of Mod-R whose objects are submodules of M-generated modules. Then $\sigma[M]$ is a locally finitely generated Grothendieck category and Theorem 1 can be applied.

Corollary 5. *Let M_R be a completely pure M-injective module which is finitely presented in $\sigma[M]$ and has a von Neumann regular endomorphism ring S. Then S is semisimple.*

Note that a module M_R which is finitely presented in $\sigma[M]$ need not be finitely presented in Mod-R; for instance, any finitely generated self-projective module (see [19] for the definition) is finitely presented in $\sigma[M]$. In particular, any simple module M is finitely presented in $\sigma[M]$, but, obviously, it need not be finitely presented in Mod-R. We also have natural interpretations of Corollaries 2 and 4 in $\sigma[M]$, with M being completely M-injective and finitely generated quasi-projective or Σ-quasi-projective (i.e., projective in $\sigma[M]$), respectively. These extend [4, Corollary 5], where the semisimplicity of $S = \mathrm{End}(M_R)$ was obtained under the stronger hypotheses that M_R is (cyclic) quasi-projective and every cyclic module in $\sigma[M]$ is M-injective.

Finally, we apply our main theorem to hereditary rings. In [2, Theorem 3.2], Colby and Rutter showed that R is a right hereditary ring with $E(R_R)$ projective iff R is a (two-sided) hereditary Artinian QF-3 ring. Clearly, if R is a ring such that $E(R_R)$ is projective, then $E(R_R)$ is finitely generated and hence finitely presented (see, e.g., [8, Lemma 13.6.6]). Thus our next result may be regarded as an extension of Colby-Rutter's theorem.

Corollary 6. *The following conditions are equivalent for a ring R:*

(1) *R is right hereditary and $E(R_R)$ is finitely presented.*
(2) *R is right hereditary right Artinian and every injective right R-module is a direct sum of finitely generated modules.*
(3) *R is a right hereditary right Artinian ring with Morita duality.*

Proof. (1) $\Rightarrow$ (2). Clearly R is right nonsingular, hence $S = \mathrm{End}(E(R_R))$ is von Neumann regular (e.g., [17, Theorem XIV.1.2]). Also, $E(R_R)$ is completely injective, so by Corollary 5, S is semisimple. It follows that $E(R_R)$, and hence R_R, have finite uniform dimension. By [15, Corollary 2], R is right Noetherian. Since $E(R_R)$ is finitely generated, it follows that R is right Artinian by [18, Theorem A]. Let $C \cong R/A$ be any cyclic right R-module. Then clearly $E(R_R)/A$ is injective and finitely generated. Thus C, being contained in $E(R_R)/A$, has a finitely generated injective hull. Now let M_R be any injective module. Then $M_R = \bigoplus_{i\in I} N_i$, where each N_i is indecomposable injective. So N_i is uniform and hence is the injective hull of any nonzero cyclic submodule $C \subseteq N_i$. This shows that N_i is finitely generated for each $i \in I$.

(2) $\Rightarrow$ (3). This follows from the well-known fact that a right Artinian ring R has Morita duality iff the injective hull of each simple right R-module is finitely generated (see, e.g., [19, 47.15]).

(3) $\Rightarrow$ (1). This is clear.

Remarks. (a) In the recent work [14], Osofsky and Smith proved a general theorem on cyclic completely CS-modules from which they obtained as a corollary the fact that right PCI-rings are right Noetherian. Corollary 2 was also obtained in [5] (for modules), by adapting the techniques of [14]. However, there are apparently no direct relationships between our Theorem 1 and the Osofsky-Smith theorem [14].

(b) The rings satisfying Corollary 6 need not be QF-3. An example of this can be found in [1, p. 353] if the division ring Δ from that example is, furthermore, assumed to be finite. However, by [10, Corollary 3.7], a right hereditary ring R with $E(R_R)$ cyclic finitely presented is semisimple. On the other hand, we do not know if Corollary 6 still remains true if the ring R is right hereditary and $E(R_R)$ is finitely generated.

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References

1. V. C. Cateforis and F. L. Sandomierski, *On modules of singular submodule zero*, Canad. J. Math. **23** (1971), 345–354.
2. R. R. Colby and E. A. Rutter, *Generalizations of QF-3 algebras*, Trans. Amer. Math. Soc. **153** (1971), 371–386.
3. R. F. Damiano, *A right PCI-ring is right Noetherian*, Proc. Amer. Math. Soc. **77** (1979), 11–14.
4. Dinh Van Huynh and R. Wisbauer, *A characterization of locally Artinian modules*, J. Algebra **132** (1990), 287–293.
5. ——, *Self-projective modules with π-injective factor modules*, J. Algebra (to appear).
6. C. Faith, *When are proper cyclics injective?*, Pacific J. Math. **45** (1973), 97–112.
7. L. Gruson and C. U. Jensen, *Dimensions cohomologiques relieés aux foncteurs* $\varprojlim^{(i)}$, Lecture Notes in Math., vol. 867, Springer-Verlag, Berlin and New York, 1981, pp. 234–294.
8. F. Kasch, *Modules and rings*, Academic Press, London, 1982.
9. B. Mitchell, *Theory of categories*, Academic Press, New York, 1965.
10. Nguyen V. Dung, *A note on hereditary rings or non-singular rings with chain condition*, Math. Scand. **66** (1990), 301–306.
11. B. L. Osofsky, *Rings all of whose finitely generated modules are injective*, Pacific J. Math. **14** (1964), 645–650.
12. ——, *Noninjective cyclic modules*, Proc. Amer. Math. Soc. **19** (1968), 1383–1384.
13. ——, *Projective dimension of 'nice' directed unions*, J. Pure. Appl. Algebra **13** (1978), 179–219.
14. B. L. Osofsky and P. F. Smith, *Cyclic modules whose quotients have all complement submodules direct summands*, J. Algebra **139** (1991), 342–354.
15. F. L. Sandomierski, *Nonsingular rings*, Proc. Amer. Math. Soc. **19** (1968), 225–230.
16. D. Simson, *On pure semi-simple Grothendieck categories*, Fund. Math. **100** (1978), 211–222.
17. B. Stenström, *Rings of quotients*, Springer-Verlag, Berlin and New York, 1975.

18. C. I. Vinsonhaler, *Orders in* QF-3 *rings*, Suppl. J. Algebra **17** (1971), 149–151.

19. R. Wisbauer, *Foundations of module and ring theory*, Gordon and Breach, Reading, MA, 1991.

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