POINCARE AND SOBOLEV INEQUALITIES IN PRODUCT SPACES

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ABSTRACT. Some local Poincaré and Sobolev inequalities involving weights in product spaces are established.

Recently there has been some interest in considering local Poincaré and Sobolev inequalities involving weights; the purpose of this note is to establish these results in the context of product spaces.

Let \( w(x, y) \) be a nonnegative locally integrable function, or weight, defined in the product space \( \mathbb{R}^n \times \mathbb{R}^m \). We say that the weight \( w \) satisfies Muckenhoupt's \( A_p(\mathbb{R}^n \times \mathbb{R}^m) \) condition, or that \( w \in A_p(\mathbb{R}^n \times \mathbb{R}^m) \), \( 1 < p < \infty \), provided that

\[
\left( \frac{1}{|P|} \int_P \int_P w(x, y) \, dx \, dy \right) \left( \frac{1}{|P|} \int_P \int_P w(x, y)^{-1/(p-1)} \, dx \, dy \right)^{(p-1)} \leq c,
\]

where \( P \) is the parallelepiped \( P = I \times J \), and \( I \subset \mathbb{R}^n \) and \( J \subset \mathbb{R}^m \) are arbitrary open cubes with sides parallel to the coordinate axes. By the Lebesgue differentiation theorem it readily follows that if \( w \in A_p(\mathbb{R}^n \times \mathbb{R}^m) \), then \( w(x, \cdot) \in A_p(\mathbb{R}^m) \) for almost every \( x \in \mathbb{R}^n \), with \( A_p \) constant independent of \( x \); similarly for \( w(\cdot, y) \).

Given a measurable set \( E \subset \mathbb{R}^n \times \mathbb{R}^m \), we denote by \( |E| \) its Lebesgue measure and \( \mu(E) = \int_E w(x, y) \, dx \, dy \). It is also convenient to introduce the notation \( \mu(x, A) = \int_A w(x, y) \, dy \) for measurable \( A \subset \mathbb{R}^m \); similarly for \( B \subset \mathbb{R}^n \) and \( \mu(B, y) \).

We say that \( w \), or \( \mu \), is doubling if

\[
\mu(2P) \leq c \mu(P), \quad 2P = 2I \times 2J, \quad \text{all } P.
\]

For \( 0 < \alpha, \beta, p, q < \infty \), we consider pairs of weights \( w, v, \) \( d\nu(x, y) = v(x, y) \, dx \, dy \), which verify

\[
\left( \frac{|I'|}{|I|} \right)^{\alpha/n} \left( \frac{|J'|}{|J|} \right)^{\beta/m} \left( \frac{\nu(P')}{\nu(P)} \right)^{1/q} \leq c \left( \frac{\mu(P')}{\mu(P)} \right)^{1/p},
\]

where \( c \) is independent of \( P' = I' \times J' \subseteq P \) and \( P \). Again by the Lebesgue differentiation theorem it follows immediately that if \( w, v \) satisfy relation (1),
then for almost every $y$ in $J$,

$$
\left( \frac{|I'|}{|I|} \right)^{\alpha/n} \left( \frac{\nu(I', y)}{\nu(I, y)} \right)^{1/q} \leq c \left( \frac{\mu(I', y)}{\mu(I, y)} \right)^{1/p}
$$

and for almost every $x$ in $I$,

$$
\left( \frac{|J'|}{|J|} \right)^{\beta/m} \left( \frac{\nu(x, J')}{\nu(x, J)} \right)^{1/q} \leq c \left( \frac{\mu(x, J')}{\mu(x, J)} \right)^{1/p}.
$$

Finally, if $f(x, y)$ is defined in an open subset of $\mathbb{R}^n \times \mathbb{R}^m$, we denote by $\nabla_1 f(x, y)$ the partial gradient of $f$ containing the $x$-derivatives; similarly for $\nabla_2 f(x, y)$, the partial gradient of $f$ containing the $y$-derivatives.

We may now state our results.

**Theorem 1 (Poincaré’s Inequality).** Assume $f$ is a Lipschitz continuous function on a parallelepiped $P$, and suppose that the weights $w, v$ satisfy the following conditions: $v$ is doubling, $w \in A_p(\mathbb{R}^n \times \mathbb{R}^m)$, and (1) holds with $\alpha + \beta < 1$ and $1 < p < \infty$. If $f_P$ denotes the average of $f$ over $P$ then

$$
\left( \frac{1}{\nu(P)} \int_P |f(x, y) - f_P|^q \, d\nu(x, y) \right)^{1/q} \leq c |I|^{1/n} \left( \frac{1}{\mu(P)} \int_P |\nabla_1 f(x, y)|^p \, d\mu(x, y) \right)^{1/p}
$$

$$
+ c |J|^{1/m} \left( \frac{1}{\mu(P)} \int_P |\nabla_2 f(x, y)|^p \, d\mu(x, y) \right)^{1/p},
$$

where $c$ is independent of $f$ and $P$.

**Theorem 2 (Sobolev’s Inequality).** Under the hypothesis of Theorem 1 and the additional assumption that $f$ is supported in $P$, we have

$$
\left( \frac{1}{\nu(P)} \int_P |f(x, y)|^q \, d\nu(x, y) \right)^{1/q} \leq c |I|^{1/n} \left( \frac{1}{\mu(P)} \int_P |\nabla_1 f(x, y)|^p \, d\mu(x, y) \right)^{1/p}
$$

$$
+ c |J|^{1/m} \left( \frac{1}{\mu(P)} \int_P |\nabla_2 f(x, y)|^p \, d\mu(x, y) \right)^{1/p},
$$

where $c$ is independent of $f$ and $P$.

**Theorem 3.** Inequality (5) holds under the hypothesis of Theorem 1 provided that $f$ vanishes on a subset $E$ of $P$ with $|E| > \eta|P|$, and now the constant $c$ depends also on $\eta$, $0 < \eta < 1$.

We pass now to the proofs, beginning with some preliminary results.

**Lemma 1.** Suppose $f$ is a Lipschitz continuous function in $P = I \times J$ and $(x, y) \in P$. Then $|f(x, y) - f_P|$ does not exceed

$$
\frac{c}{|P|} \int_P \left( |\nabla_1 f(u, z)||u - x| + |\nabla_2 f(u, z)||z - y| \right) \times \min \left( \frac{|I|^{1/n}}{|u - x|}, \frac{|J|^{1/m}}{|z - y|} \right)^{n + m} \, du \, dz,
$$

where $c$ is a constant independent of $f$ and $P$. 

Proof. It is clear that \(|f(x, y) - f_P| \leq A + B\), say, where
\[
A = \frac{1}{|P|} \int_P \int_0^1 \int_0^1 |\nabla_1 f(x + t(u - x), y + t(z - y))||u - x||dtdudz,
\]
and the expression for \(B\) is obtained by replacing \(\nabla_1\) by \(\nabla_2\) above; since both integrals are handled in a similar fashion we only consider \(A\). If \(\chi_P\) denotes the characteristic function of \(P\) then we may rewrite \(A\) as
\[
\frac{1}{|P|} \int_P \int_0^1 |\nabla_1 f(u, z)||u - x| \int_0^1 \chi_P \left(x + \frac{u - x}{t}, y + \frac{z - y}{t}\right) t^{n-m-1} dt dudz.
\]
Furthermore, since the integrand in the innermost integral above vanishes if either \(|u - x| \geq t|\xi_x|\) or \(|z - y| \geq t|\xi_y|\), it readily follows that
\[
A \leq \frac{1}{|P|} \int_P \int_0^1 \int_0^\infty \int_{\max(|u - x|/|\xi_x|, |z - y|/|\xi_y|)}^\infty \frac{t^{n-m-1} dt dudz}{|u - x| |z - y|} < \frac{1}{(n + m)|P|} \int_R \int_R |\nabla_1 f(u, z)||u - x| \min \left(\frac{|I|^{1/n}}{|u - x|}, \frac{|J|^{1/m}}{|z - y|}\right)^{n+m} du dz,
\]
and the proof is complete. □

Corollary 1. Let \(0 < \gamma, \lambda < 1\). Under the hypothesis of Lemma 1 we also have
\[
|f(x, y) - f_P| \leq c \left(\frac{|I|^{1/n}}{|J|^{1/m}}\right)^\gamma \int_P \int_{\max(|u - x|/|\xi_x|, |z - y|/|\xi_y|)}^\infty \frac{|\nabla_1 f(u, z)||u - x|} t^{n-m-1} dudz,
\]
where, if \(\psi\) denotes a nonnegative compactly supported smooth function on \(R^1\) and \(\psi_t(x) = t^{-\gamma}\psi(|x|/t)\) and \(\psi_s(y) = s^{-\gamma}\psi(|y|/s)\), then
\[
\int_{R^n} \int_{R^n} g(u, z) \frac{dudz}{|u - x|^{n-(1-\gamma)}|z - y|^{m-\gamma}} = \int_0^\infty \int_0^\infty t^{1-\gamma} s^{1-\gamma} G(x, t, y, s) dt ds,
\]
where, if \(c\) denotes a constant that only depends on \(\psi\),
\[
G(x, t, y, s) = c \int_{R^n} \int_{R^n} g(u, z) \psi_t(x - u) \psi_s(y - z) dudz.
\]
Thus, to estimate the innermost integral in (8), we set
\[ g(u, z) = |\nabla f(u, z)| \chi_R(u, z) \]
in the above expression, and breaking up the domain of integration into four parts, namely, 
\([0, |I|^{1/n}) \times [0, |J|^{1/m}), [0, |I|^{1/n}) \times [|I|^{1/m}, \infty), [|I|^{1/n}, \infty) \times [0, |J|^{1/m}), \text{and}\ [|I|^{1/n}, \infty) \times [|J|^{1/m}, \infty), \]
we obtain that (8) is bounded by the sum of four terms, \(A_1 + A_2 + A_3 + A_4\), say, where \(A_1\) is equal to
\[
\left( \frac{|I|^{1/n}}{|J|^{1/m}} \right)^q \left( \int_0^1 \int P \left( \int_0^{[|I|^{1/m}} \int_0^{[|I|^{1/n}} t^{(1-\gamma)-1}s^{q-1} \times G(x, t, y, s)\, dt\, ds \right)^q \nu(x, y) \right)^{1/q},
\]
and where \(A_2, A_3,\) and \(A_4\) are defined similarly.

It is easy to estimate \(A_4\); indeed, since \(\psi(u) \leq Ct^{-n}\) and \(\psi(z) \leq Cs^{-m}\), it readily follows that \(A_4 \leq c|I|^{1/n}\|g\|_1\), which, by the \(A_p(R^n \times R^m)\) condition, is a bound of the right order.

We turn now to estimate \(A_1\). For \((x, t)\) a point in \([0, |I|^{1/n})\), consider the integral
\[
I(x, t) = \int_0^{[|I|^{1/m}} \int J(x, t, y, s) v(x, y)\, dy\, ds,
\]
and observe that if \(d\nu_q(x, y, s) = v(x, y)s^{q-1}\) then \(I(x, t) = \int_0^{[|I|^{1/m}} \int J(x, t, y, s) v(x, y)\, dy\, ds\), and observe that if \(d\nu_q(x, y, s) = v(x, y)s^{q-1}\) then \(I(x, t) = \int_0^{[|I|^{1/m}} \int J(x, t, y, s) v(x, y)\, dy\, ds\), \(U(x, t, \lambda) = \{(y, s) \in J \times [0, |J|^{1/m}) : G(x, t, y, s) > \lambda, \}\), then
\[
I(x, t) = q \int_0^{[|I|^{1/m}} \lambda^{q-1} \nu_q(U(x, t, \lambda))\, d\lambda.
\]

Now let
\[
NG(x, t, y) = \sup_{|y-z|<s} G(x, t, z, s),
\]
and for \(\lambda > 0\), put
\[
\Theta = \{y \in J : NG(x, t, y) > \lambda\}.
\]

By the Whitney decomposition there is a sequence \(\{J_k\}\) of nonoverlapping closed cubes, subcubes of \(J\) actually, such that \(\Theta = \bigcup_k J_k\) and
\[
U \subseteq \bigcup_k (J_k \times [0, C|J_k|^{1/m})),
\]
where \(C\) is a dimensional constant. Whence,
\[
\nu_q(U(x, t, \lambda)) \leq \sum_k \nu_q(J_k \times [0, C|J_k|^{1/m})),
\]
\[
= \sum_k \int_0^{C|J_k|^{1/m}} s^{q-1} \nu(J_k, v(x, y)\, dy = c \sum_k |J_k|^{q/q'\nu(J_k)},
\]
\[
\leq c|J|^{q/q'\nu(J)} \sum_k \left( \frac{|J_k|}{|J|} \right)^{q/q'\nu(J)} \left( \frac{v(x, J_k)}{v(x, J)} \right).
\]
We may now invoke the estimate in (3) and dominate the above expression by

$$c|J|^q \beta/m \nu(x, J) \sum_k \left( \frac{\mu(x, J_k)}{\mu(x, J)} \right)^{q/p}$$

(11)

$$\leq c|J|^q \beta/m \nu(x, J) \mu(x, J)^{-q/p} \mu(x, \mathcal{O}(x, t, \lambda))^{q/p}.$$

Substituting (11) into (10) gives

$$I(x, t) \leq c|J|^q \beta/m \nu(x, J) \mu(x, J)^{-q/p} \int_0^\infty \lambda^{q-1} \mu(x, \mathcal{O}(x, t, \lambda))^{q/p} d\lambda.$$

Next consider the integral

$$B = \int_0^{|I|/n} \int_0^1 I(x, t) \, dx \, dt \leq c|J|^q \beta/m \int_0^\infty \lambda^{q-1} R(\lambda) \, d\lambda,$$

where

(12)

$$R(\lambda) = \int_0^{|I|/n} \int_0^1 \left( \frac{\mu(x, \mathcal{O}(x, t, \lambda))}{\mu(x, J)} \right)^{q/p} \nu(x, J) \lambda^{q-1} \, dx \, dt.$$

Observe that if $F(x, t, \lambda) = \mu(x, \mathcal{O}(x, t, \lambda))/\mu(x, J) \leq 1$ and $d\mu_q(x, t) = \nu(x, J) \lambda^{q-1} \, dx \, dt$, then we may write $R(\lambda)$ as

(13)

$$\frac{q}{p} \int_0^1 \zeta^{q/p-1} \mu_q(\{|(x, t) \in I \times [0, |I|/n]: F(x, t, \lambda) > \zeta\}) \, d\zeta.$$

In order to estimate (14), once again we introduce appropriate maximal functions, namely,

$$NG(x, y) = \sup_{|x-u|<t, \ |y-z|<s} G(u, t, z, s)$$

and

$$NF(x, \lambda) = \sup_{|x-u|<t} \chi_I(u)F(u, t, \lambda) \leq 1.$$

Let $\mathcal{O}_\zeta$ be the open set $\{NF(x, \lambda) > \zeta\} \cap I$; $\mathcal{O}_\zeta \neq \emptyset$ only for $\zeta \leq 1$. According to the Whitney decomposition there is a sequence $\{I_k\}$ of nonoverlapping cubes so that $\mathcal{O}_\zeta = \bigcup_k I_k$, and if $\mathcal{U}(\lambda, \zeta) = \{|(x, t) \in I \times [0, |I|/n]: F(x, t, \lambda) > \zeta\}$ then $\mathcal{U}(\lambda, \zeta) \subseteq \bigcup_k (I_k \times [0, C|I|^{1/n}]).$ Thus, by (1),

$$\mu_q(\mathcal{U}(\lambda, \zeta) \leq \sum_k \mu_q(I_k \times [0, C|I|^{1/n}])$$

(15)

$$\leq c|I|^{q\alpha/n} \nu(P) \sum_k \left( \frac{|I_k|}{|I|} \right)^{q\alpha/n} \left( \frac{\nu(I_k \times J)}{\nu(P)} \right)^{q/p}$$

$$\leq c|I|^{q\alpha/n} \nu(P) \sum_k \left( \frac{\mu(I_k \times J)}{\mu(P)} \right)^{q/p}.$$

Whence substituting (15) into (14), we immediately get

(16)

$$R(\lambda) \leq c \frac{|I|^{q\alpha/n} \nu(P)}{\nu(P)q/p} \int_0^1 \zeta^{q/p-1} \left( \sum_k \mu(I_k \times J) \right)^{q/p} d\zeta.$$
Next we estimate the integral in (16). The sum there does not exceed 
\( \mu(\zeta' \times J) \). Furthermore, since \( \mathcal{O}(x, t, \lambda) \subseteq \{y \in J : NG(x, y) > \lambda\} \), it readily follows that
\[
F(x, t, \lambda) \leq \frac{\mu(x, \{y \in J : NG(x, y) > \lambda\})}{\mu(x, J)}.
\]

Thus,
\[
\mathcal{O}_\zeta' \subset \mathcal{U}(\lambda, \zeta) = \{x \in I : \mu(x, \{y \in J : NG(x, y) > \lambda\}) > \zeta \mu(x, J)\},
\]
and the integral in (16) is bounded by
\[
\int_0^1 \zeta^{q/p-1} \left( \int_J \int_I \mathcal{X}_{\mathcal{U}(\lambda, \zeta)}(x) w(x, y) \, dx \, dy \right)^{q/p} \, d\zeta = \int_0^1 \zeta^{q/p-1} \, g(\zeta)^{q/p} \, d\zeta,
\]
say. Moreover, since \( g(\zeta) \) decreases with \( \zeta \), it is clear that the last integral above does not exceed
\[
(17) \quad c \left( \int_0^1 g(\zeta) \, d\zeta \right)^{q/p} = c \left( \int_J \int_0^1 \mathcal{X}_{\mathcal{U}(\lambda, \zeta)}(x) \mu(x, J) \, d\zeta \, dx \right)^{q/p}.
\]

Setting \( \zeta' = \zeta \mu(x, J) \), it readily follows that the innermost integral in (17) is bounded by \( \mu(x, \{y \in J : NG(x, y) > \lambda\}) \), and, consequently, the expression appearing in (17) does not exceed \( c \mu(\{(x, y) \in P : NG(x, y) > \lambda\})^{q/p} \). Substituting this into (16) gives
\[
R(\lambda) \leq c |I|^{q_0/n} \nu(P) \left( \frac{\mu(\{(x, y) \in P : NG(x, y) > \lambda\})}{\mu(P)} \right)^{q/p},
\]
which in turn implies that the integral \( B \) in (12) is less than or equal to
\[
(18) \quad c |I|^{q_0/n} |J|^{q_1/m} \nu(P) \left( \frac{\nu(P)}{\mu(P)} \right)^{q/p} \int_0^1 \lambda^{q-1} \mu(\{(x, y) \in P : NG(x, y) > \lambda\})^{q/p} \, d\lambda
\]

Finally we are ready to estimate \( A_1 \). Let \( 0 < \epsilon = \alpha/(1 - \gamma) \), \( \delta = \beta/\gamma < 1 \), and observe that by Hölder’s inequality the integral in (9) is bounded by
\[
\int \int_P \left( \int_0^{|J|^{1/m}} \int_0^{|I|^{1/n}} t^{(1-\gamma)(1-\epsilon)q'-1}s^{\gamma(1-\delta)q'-1} \, dt \, ds \right)^{q/q'} \times \left( \int_0^{|J|^{1/m}} \int_0^{|I|^{1/n}} t^{(1-\gamma)q'-1}s^{\delta q'-1} G(x, t, y) \, dt \, ds \right) \, d\nu(x, y)
\]
\[
= c |I|^{q(1-\gamma-\alpha)/m} |J|^{q(1-\gamma)/m} \times \int \int_P \int_0^{|J|^{1/m}} \int_0^{|I|^{1/n}} t^{\alpha-1}s^{\delta q'-1} G(x, t, y) \, dt \, ds \, d\nu(x, y).
\]
Now, using estimate (18) for the above integral, be well-known properties of $A_p(R^n \times R^m)$ weights, it follows at once that

$$A_1 \leq c |I|^{1/n} \left( \frac{1}{\mu(P)} \int \int_{p} NG(x, y)^p \, d\mu(x, y) \right)^{1/p}$$

$$\leq c |I|^{1/n} \left( \frac{1}{\mu(P)} \int \int_{R^n \times R^m} \chi_P(x, y) |\nabla f(x, y)|^p \, d\mu(x, y) \right)^{1/p},$$

which is a bound of the right order.

To handle $A_2$, let

$$H(x, t) = \int \int_{I} |\nabla f(u, y)| \, \eta_t(x - u) \, du \, dy.$$

Clearly

$$(19) \quad A_2 \leq c \left( \frac{|I|^{1/n}}{|I|^{3/n} |J|} \right)^{1/q} \left( \frac{1}{\nu(P)} \int_0^{1/n} \int_0^{1/m} \int_0^{t^{q-1}} H(x, t)^q \, d\nu(x, y) \, dt \right)^{1/p}.$$  

In order to estimate the integral in (19), let $C(I) = \{(u, t) : u \in I, \, 0 < t < |I|^{1/n}\}$, define

$$NH(x) = \sup_{|x-u|<t} \chi_{C(I)}(u, t) H(x, t),$$

and put

$$\mathcal{V}(\lambda) = \{(u, t) \in C(I) : H(u, t) > \lambda\}.$$

If $d\nu_q(x, y, t) = \nu(x, y) t^{q-1} \, dx \, dy \, dt$ then the integral in (19) is dominated by $q \int_0^{\infty} \lambda^{q-1} \nu_q(\mathcal{V}(\lambda) \times J) \, d\lambda$, and, consequently, by a familiar argument we also have

$$A_2 \leq c |I|^{1/n} |J|^{1/m} \left( \frac{1}{\mu(P)} \int \int_{p} \left( \int \int |\nabla f(x, z)| \, dz \right)^p \, d\mu(x, y) \right)^{1/p}.$$  

Now, since $w \in A_p(R^n \times R^m)$, this bound is also of the right order. $A_3$ is treated in an analogous fashion, and the proof is complete. □

Proof of Theorems 2 and 3. The proof of these results is similar to that of Theorem 1. In fact, if $f$ is defined on $P$ and vanishes on some subset $E$ of $P$ with $|E| > \eta |P|$, then for $(x, y) \in P$,

$$|f(x, y)| \leq |f(x, y) - f_P| + \frac{1}{|E|} \int \int_{E} |f(u, z) - f_P| \, du \, dz$$

$$\leq |f(x, y) - f_P| + \frac{1}{\eta |P|} \int \int_{P} |f(u, z) - f_P| \, du \, dz.$$  

Whence, by Corollary 1, for $(x, y) \in P$, $|f(x, y)|$ is also dominated by the right-hand side in estimate (7), and Theorem 3 has been proved.

If, on the other hand, $f$ is compactly supported and its support is contained in $P$, then we may extend $f$ to be 0 off $2P$, say, and establish Theorem 2 from Theorem 3. □
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