LOGARITHMIC CONVEXITY OF PERRON-FROBENIUS EIGENVECTORS OF POSITIVE MATRICES

SIDDHARTHA SAHI

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Abstract. Let $C(S)$ be the cone of Perron-Frobenius eigenvectors of stochastic matrices that dominate a fixed substochastic matrix $S$. For each $0 \leq \alpha \leq 1$, it is shown that if $u$ and $v$ are in $C(S)$ then so is $w$, where $w_j = u^\alpha v_j^{1-\alpha}$.

The basic result of Perron-Frobenius theory [S] is that if a matrix has strictly positive entries, then its maximal eigenvalue is unique, positive, occurs with multiplicity 1, and has a (coordinatewise) positive eigenvector.

Subsequent literature on positive matrices contains many results (e.g., [C, F, K]) that deal with convexity properties of the dominant eigenvalue as a function of matrix entries. Similar results for the corresponding eigenvectors are obtained in [DN, EJN] but only for the effects of varying a single row of the matrix. Little seems to be known about the behavior of these eigenvectors under a more general perturbation of the matrix.

In this paper we prove a different kind of convexity property for Perron-Frobenius eigenvectors that was motivated by economic considerations in [SY] but that, by virtue of its unexpected and elementary nature, seems to warrant a wider mathematical audience.

For convenience, we formulate the result in terms of stochastic matrices—a positive matrix is called stochastic (substochastic) if its column sums are equal to (less than) 1.

For a fixed substochastic matrix $S$, consider the cone $C(S)$ of all (positive) Perron-Frobenius eigenvectors of the various stochastic matrices that (entrywise) dominate $S$. Thus $C(S) = \{v > 0 \mid \exists$ stochastic $A \succeq S$ such that $Av = v\}$.

Now $C(S)$ need not be a convex subset of $\mathbb{R}^n$. However, we shall show that it has the following remarkable property that may be termed logarithmic, or geometric, convexity.

Theorem. Fix $0 \leq \alpha \leq 1$, and put $\beta = 1 - \alpha$. If $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ are in $C(S)$ then so is $w = (w_1, w_2, \ldots, w_n)$ where $w_j = u^\alpha v_j^{\beta}$.
The principal difficulty in proving this theorem is the indirect nature of the definition of $C(S)$. The following lemma "eliminates the quantifier" in that definition.

**Lemma.** A positive vector $v$ belongs to $C(S)$ if and only if $Sv \leq v$.

**Proof.** If $v$ is in $C(S)$, choose $A \geq S$ such that $v = Av$. Then clearly $Sv \leq Av = v$.

Conversely, suppose $v > 0$ with $Sv \leq v$, and put $\delta = v - Sv$. Also, let $s_j$ be the $j$th column sum of $S$, and put $\epsilon_j = 1 - s_j$. Clearly, $0 < \epsilon_1 v_1 + \cdots + \epsilon_n v_n = \delta_1 + \cdots + \delta_n = \lambda$, say. Now let $A$ be the matrix whose $ij$th entry is $a_{ij} = s_{ij} + \frac{1}{\lambda} \delta_i \epsilon_j$. It is easily checked that $A$ is stochastic, dominates $S$, and satisfies $Av = v$. □

**Proof of Theorem.** In view of the lemma, we may assume that $Su \leq u$ and $Sv \leq v$, and we have to show that $Sw \leq w$. Using the Hölder inequality, we get

\[
(Sw)_i = \sum_j s_{ij} w_j = \sum_j s_{ij} u_j^\alpha v_j^\beta = \sum_j (s_{ij} u_j)^\alpha (s_{ij} v_j)^\beta \\
\leq \left( \sum_j s_{ij} u_j \right)^\alpha \left( \sum_j s_{ij} v_j \right)^\beta \leq (u_i)^\alpha (v_i)^\beta = w_i. \quad \square
\]

**References**


