ON PRIME IDEALS IN RINGS OF SEMIALGEBRAIC FUNCTIONS

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Abstract. It is proved that if \( p \) is a prime ideal in the ring \( S(M) \) of semi-algebraic functions on a semialgebraic set \( M \), the quotient field of \( S(M)/p \) is real closed. We also prove that in the case where \( M \) is locally closed, the rings \( S(M) \) and \( P(M) \)—polynomial functions on \( M \)—have the same Krull dimension. The proofs do not use the theory of real spectra.

Let \( R \) be a real closed field. The only topology we consider in \( R^n \) is the euclidean topology. For every semialgebraic subset \( M \subset R^n \) we denote by \( S(M) \) the ring of semialgebraic functions on \( M \), i.e., continuous maps \( f: M \to R \) whose graph is a semialgebraic subset of \( R^{n+1} \), and \( P(M) \) stands for the ring of polynomial functions on \( M \), i.e., restrictions to \( M \) of polynomials in \( R[x_1, \ldots, x_n] \). In this note we give elementary proofs of the following results.

**Theorem 1.** For every prime ideal \( p \) of \( S(M) \), the quotient field of \( S(M)/p \) is real closed.

**Theorem 2.** Let \( \pi: \text{Spec} S(M) \to \text{Spec} P(M) \) be the map induced by the inclusion \( P(M) \hookrightarrow S(M) \), and assume that \( M \) is locally closed. Then:

1. The fibers of \( \pi \) are \( T_1 \)-spaces.
2. The Krull dimensions of \( P(M) \) and \( S(M) \) are equal.
3. If \( M \subset R^n \) is algebraic, the image of \( \pi \) is the set \( \text{Spec}^r(M) \) of real prime ideals of \( P(M) \).

For every ring \( A \), \( \text{Spec} A \) denotes the prime spectrum of \( A \) endowed with its Zariski topology. The ideal \( a \) in \( A \) is real if whenever \( f_1, \ldots, f_k \in A \) and \( f_1^2 + \cdots + f_k^2 \in a \), then \( f_1, \ldots, f_k \in a \). Theorem 1 can be deduced from [9, Corollary 3.26, §1, and Theorem 1.1]. Schwartz's proof involves his theory of real closed spaces. This can be viewed as the semialgebraic counterpart of results by Henriksen and Isbell [4] and Isbell [5]. In fact, if \( m \) is a maximal ideal in the ring of real-valued continuous functions on a normal topological space \( X \), it is proved in [4] that the quotient \( C(X)/m \) is a real closed field. In [5], the normality condition on \( X \) is dropped. The second part of Theorem 2

Let us fix some notation. For each ideal $a$ in $S(M)$, the set of common zeros in $M$ of functions in $a$ is denoted by $Z(a)$. If $a = f \cdot S(M)$ for some $f \in S(M)$ we abbreviate $Z(f) = Z(f \cdot S(M))$. Given a subset $X \subset M$, the ideal of all functions $f \in S(M)$ such that $X \subset Z(f)$ is denoted by $\mathcal{J}(X)$, while $\overline{X}$ is the smallest algebraic subset of $R^n$ containing $X$. Finally, for each $f \in S(M)$ we denote by $|f| \in S(M)$ the "absolute value of $f$".

**Proof of Theorem 1.** Let $A = S(M)/p$ and let $E = q \cdot f(A)$ be its quotient field. First we must prove that either $x$ or $-x$ is a square in $E$ for every $x \in E$. Write $x = (h + p)(g + p)^{-1}$, $f, g \in S(M)$ and $g \notin p$. If $h = fg$ we get $(|h| - h)(|h| + h) = 0 \in p$, and so either $|h| - h \in p$ or $|h| + h \in p$. In the first case, $x = (h + p)(g + p)^{-2} = (|h| + p)(g + p)^{-2}$ is a square in $E$, since $|h|$ is a square in $S(M)$. Analogously, $-x$ is a square in $E$ in the second case. Hence, we must only show that each odd degree polynomial $P \in E[T]$ has at least one root in $E$. We may suppose that $P \in A[T]$ is monic. In fact, if we write $P(T) = (a_0 T^m + a_1 T^{m-1} + \cdots + a_m) \cdot b^{-1}$, $a_1 \in A$, $a_0, b \in A \setminus \{0\}$, then we construct the monic polynomial $Q(T) = T^m + \sum_{j=1}^m a_j \cdot a_0^{-1} T^{m-j} \in A[T]$, and if $a \in E$ is a root of $Q$, then $a \cdot a_0^{-1} \in E$ is a root of $P$. Thus, from now on we put $P(T) = T^m + \sum_{j=1}^m f_j T^{m-j}$, $f_j \in S(M)$, and $m$ is odd. Let us consider the polynomial $F_0 = T^m + \sum_{j=1}^m x_j T^{m-j} \in R[x_1, \ldots, x_m, T]$. Clearly, $F_0$ and its derivatives $F_j = \partial F_0 / \partial T^j$, $1 \leq j \leq m$, are monic (modulo factors in $N$) with respect to $T$, and the family $\mathcal{F} = \{F_0, F_1, \ldots, F_m\}$ is stable under derivation (with respect to $T$). Therefore, if $(A_i; \zeta_{ij}) : 1 \leq i \leq k$, $1 \leq j \leq k(i)$) is a "saucissonage" of $\mathcal{F}$ (see [1, 2.3.4], then functions $\zeta_{ij} \in S(A_i)$ can be extended to the closure $A_i$, by [1, 2.5.6] and so, using the semialgebraic Tietze's extension theorem [1, 2.6.10], there exist semialgebraic functions $\eta_{ij} \in S(R^m)$ such that $\eta_{ij}$ restricted to $A_i$ coincides with $\zeta_{ij}$. By the very definition of "saucissonage", and since $m$ is odd, there exists for every $1 \leq i \leq k$ an index $l(i) \in \{1, \ldots, k(i)\}$ such that $F_0(x, \eta_{l(i), l(i)}(x)) = 0$ for each point $x \in A_i$.

On the other hand, if $\varphi = (f_1, \ldots, f_m) : M \to R^m$, the compositum $g_i = \eta_{l(i), l(i)} \circ \varphi$ belong to $S(M)$, and all reduce to proving that $g_i + p$ is a root of $P$ for some $i$. For every point $y \in M$, $x = \varphi(y) \in R^m = A_1 \cup \cdots \cup A_k$ and so

$$\prod_{j=1}^k F_0(\varphi(y), g_j(y)) = \prod_{i=1}^k F_0(x, \eta_{l(i), l(i)}(x)) = 0.$$

Consequently, the polynomial $H(T) = T^m + \sum_{j=1}^m f_j \cdot T^{m-1}$ verifies that the product $H(g_1) \cdots H(g_k) = 0$, since for each $y \in M$,

$$F_0(\varphi(y), g_i(y)) = g_i(y)^m + \sum_{j=1}^m f_j(y) g_i^{m-j}(y).$$

Finally, since $p$ is prime, $H(g_i) \in p$ for some $1 \leq i \leq k$, i.e., $P(g_i + p) = 0$.

**Proof of Theorem 2.** (1) We must prove that $a \cap P(M) \subsetneq b \cap P(M)$ for given prime ideals $a \subsetneq b$ in $S(M)$. Let us take $f \in b \setminus a$. Its zero-set $Z(f)$ is a
closed semialgebraic subset of $M$ and so, by the finiteness theorem (see [6]
or [1, 2.7.1]), there exist polynomial functions $f_{ij} \in P(M)$ such that
$Z(f) = \bigcup_{i=1}^{m} \{x \in M : f_{i1}(x) \geq 0, \ldots, f_{ik}(x) \geq 0\}$. Define
$h_i = \sum_{j=1}^{k} (f_{ij} - |f_{ij}|)^2$, $i = 1, \ldots, m$. Then $Z(f) = Z(h)$ for $h = h_1 \cdots h_m$, and so $\mathcal{I}(Z(f)) = \mathcal{I}(Z(h))$. Hence, by the semialgebraic Nullstellensatz [1, 2.6.7], $\sqrt{h} \cdot S(M) = \sqrt{h} \cdot S(M)$. In particular, since $f \notin a$, we conclude that every $h_i \notin a$ and, from $f \in b$, also $h_i \notin b$ for some $1 \leq i \leq m$. From Theorem 1, the quotient field of $S(M)/b$ is formally real, i.e., $b$ is a real ideal, and $h_i \notin b$ for some $1 \leq i \leq m$. Thus, $f_{ij} - |f_{ij}| \in b$ for all $1 \leq j \leq k$, and since $h_i \notin a$, there exists $j$ with $f_{ij} - |f_{ij}| \in b \setminus a$. To finish we shall check that $g = f_{ij} \in [b \cap P(M)]/[a \cap P(M)]$. In fact, $0 = (g - |g|) \cdot (g + |g|)$ and since $a$ is prime, $g + |g| \in a \subset b$. Thus, $g = [(g - |g|) + (g + |g|)] \cdot 2^{-1} \in b$.

On the other hand, if $g \in a$, then $|g|^2 = g^2 \in a$ also, i.e., $|g| \in a$, which implies $g - |g| \in a$, absurd.

(2) The inequality $\dim S(M) \leq \dim P(M)$ is a consequence of part (1). Let $d = \dim P(M) = \dim M$. Then $M$ contains a closed semialgebraic subset $K$ semialgebraically homeomorphic to the cube $I = [-1, 1]^d \subset R^d$. From Tietze’s theorem [1, 2.6.10], $\dim S(M) \geq \dim S(K) = \dim S(I)$ and so it suffices to see that $d \leq \dim S(I)$. In the polynomial ring $A = R[x_1, \ldots, x_d]$ we consider the chain of prime ideals

$$(0) = p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_d; \quad p_k = (x_1, \ldots, x_k) \cdot A.$$ 

The quotient fields $E_k = q \cdot f(A/p_k) \approx R(x_{k+1}, \ldots, x_d)$ are formally real, and each ordering in $E_k$ can be extended to $E_{k-1}$. So we can choose cones $\alpha_k$ of nonnegative elements in $E_k$ such that $(E_k, \alpha_k)$ is an ordered extension of $(E_{k+1}, \alpha_{k+1})$. Now define the ideals

$q_k = \{f \in S(I) : \text{there exists } g_1, \ldots, g_l \in A \text{ such that } g_i + p_k \in \alpha_k \text{ and } P(g_1, \ldots, g_l) = \{x \in I : g_1(x) \geq 0, \ldots, g_l(x) \geq 0\} \subset Z(f)\}.$

Obviously $q_0 \subset q_1 \subset \cdots \subset q_d$ and so it is enough to check that $q_k$ is prime and $q_k \cap A = p_k$. In what follows $\overline{g}$ denotes the class mod $p_k$ of $g \in A$. Let $f, g \in S(I)$ such that $fh \in q_k$. Then $Z(f) \cup Z(h)$ contains the set $P(g_1, \ldots, g_l)$ for some $g_1, \ldots, g_l \in A$ with $\overline{g_i} \in \alpha_k$. From the finiteness theorem [6] we can write $Z(f) = \bigcup_{i=1}^{m} P(f_{i1}, \ldots, f_{il})$, $Z(h) = \bigcup_{i=1}^{m} P(h_{i1}, \ldots, h_{il})$ for certain $f_{ij}, h_{ij} \in A$ and in case neither $f$ nor $h$ belong to $q_k$, there exists a family $\{f_{i,j(i)}, h_{i,j(i)} : 1 \leq i \leq m\}$ such that $\overline{f_{i,j(i)}} \in \alpha_k$, $\overline{h_{i,j(i)}} \notin \alpha_k$. From Artin-Lang theorem [1, 4.1.2] there exists a homomorphism $\mathcal{I} : A/p_k \to R$ such that:

$$(i) \mathcal{I}(\overline{g}) \geq 0; \quad (ii) \mathcal{I}(\overline{f}) < 0; \quad (iii) \mathcal{I}(\overline{h}) < 0; \quad (iv) p = (\mathcal{I}(\overline{x_1}), \ldots, \mathcal{I}(\overline{x_d})) \in I.$$ 

Then, each $g_i(p) = \mathcal{I}(\overline{g}) \geq 0$ and so $p \in Z(f) \cup Z(h)$ which is false since $f_{ij(i)}(p) = \mathcal{I}(\overline{f_{ij(i)}}) < 0$ and $h_{ij(i)}(p) < 0$ for all $i$. Hence $q_k$ is prime. Also, for $f \in p_k$ we have $Z(f) = P(f, -f)$ and $\overline{f}, -\overline{f} \in \alpha_k$, and so $p_k \subset q_k \cap A$. Finally, if some $f \in q_k \cap A$ exists, but $f \notin p_k$, then $Z(f) \supset P(g_1, \ldots, g_l)$ for some $g_i \in A$ with $\overline{g_i} \in \alpha_k$. Again from the Artin-Lang theorem we get a homomorphism $\psi : A/p_k \to R$ such that $\psi(f) \neq 0$, $\psi(\overline{g}) \geq 0$, and $q = (\psi(\overline{x_1}), \ldots, \psi(\overline{x_d})) \in I$, i.e., $q \in P(g_1, \ldots, g_l) \setminus Z(f)$, which is absurd.
Each prime ideal in $S(M)$ is real. Hence $\text{Spec}^r P(M)$ contains the image of $\pi$. For the converse, assume first that $M$ is irreducible and $p = p_0$ is the zero ideal in $P(M)$. Let $a \in M$ be a regular point of dimension $d = \dim M$ of $M$, and let $U$ be an open neighborhood of $a$ in $\mathbb{R}^n$ such that there exists a semialgebraic homeomorphism $F: \Delta_d = [-1, 1]^d \to M \cap U$ with $F(0) = a$. For every $\varepsilon \in R^+$ let us denote $A_\varepsilon = \{x \in \Delta_{d-1} : 0 < x_i < \varepsilon, i = 1, \ldots, d-1\}$. For every semialgebraic function $\mathcal{S}_x: A_\varepsilon \to R^+$, define

$$A_\varepsilon(\mathcal{S}) = \{(x', x_d) \in \mathbb{R}^d : x' \in A_\varepsilon \text{ and } 0 < x_d < \mathcal{S}(x')\}.$$ 

Then we construct a prime ideal in $S(M)$ as follows: $q = \{h \in S(M) : \text{there exists } \varepsilon \in R^+ \text{ and a semialgebraic function } \mathcal{S}_x: \Delta_\varepsilon \to R^+ \cup \{0\} \text{ with } Z(\mathcal{S}) = \{a\} \text{ such that } (f|\Delta \cap U) \circ F \text{ vanishes on } A_\varepsilon\}$. Moreover, $q \cap P(M) = p_0$ since every $f \in q \cap P(M)$ vanishes on $M \cap U$ and so over $\Delta \cap U^2 = M$.

If $M$ is irreducible and $p$ is an arbitrary prime ideal in $P(M)$, the zero set $N = Z(p) \subset M$ is an irreducible algebraic set and so there exists a prime ideal $q_N$ of $S(N)$ lying over the zero ideal of $P(N)$. Let $r^*: \text{Spec} S(N) \to \text{Spec} S(M)$ be the map induced by the restriction homomorphism $r: S(M) \to S(N)$. Then $q = r^*(q_N)$ verifies $q \cap P(M) = p$, by the real Nullstellensatz [1, 4.4.3].

Finally let $M$ be arbitrary with irreducible components $M_1, \ldots, M_k$ and let $p$ be a real prime ideal in $P(M)$. Write $A = R[x_1, \ldots, x_n]$ and $I$ (resp. $I_i$) the ideal of polynomials in $A$ vanishing on $M$ (resp. $M_i$). There exists a prime ideal $p^*$ in $A$ containing $I = I_1 \cap \cdots \cap I_k$ such that $p = p^*/I$. We can suppose that $p^*$ contains $I_1$ and so $p_1 = p^*/I_1$ is a real prime ideal in $P(M)$. Hence there exists a prime ideal $q_1$ in $S(M_1)$ such that $q_1 \cap P(M_1) = p_1$ and so, if $r_1: S(M) \to S(M_1)$ is the restriction homomorphism, we get $q = r_1^*(q_1) \in \text{Spec} S(M)$ such that $q \cap P(M) = p$.

**Remark.** Part (3) of Theorem 2 is no longer true for more general semialgebraic subsets $M \subset \mathbb{R}^n$. Consider for example a nonfinite semialgebraic subset $M$ of $R$, $M \neq R$, a point $a \in R \setminus M$, and the function $f: M \to R$ defined by $f(x) = x - a$. Then $a = f \cdot P(M)$ is a real ideal but since $Z(f)$ is empty there is no prime ideal in $S(M)$ lying over $p$.

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**References**


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