BANACH ALGEBRAS WHICH ARE NOT WEDDERBURNIAN

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Abstract. Let $A$ be a Banach algebra with radical $R$. In 1951 Feldman exhibited an example in which it is impossible to find a closed subalgebra $K$ of $A$ such that $A = K \oplus R$. We provide other examples. Feldman's algebra is commutative, but these examples are, in general, not commutative.

1. Introduction

In [5, p. 85] Glaeser called a Banach algebra $A$ Wedderburnian if $A$ is the direct sum of its radical $R$ and a closed subalgebra $K$ of $A$. In [2] Bade and Curtis called such a Banach algebra strongly decomposable. If $A$ is finite dimensional then a classical result of Wedderburn shows that $A$ is Wedderburnian. In [4] Feldman provided an example where $A$ is not Wedderburnian. This algebra was studied in detail in [1]. For another example see [5]. These examples are commutative. Our aim is to provide noncommutative examples which occur rather naturally. Some instances arise as follows. Let $G$ be an infinite compact topological group with identity $e$. Let $C(G)$ be the set of all complex-valued continuous functions on $G$ taken as an algebra with convolution as its multiplication. Let $\|f\|_2$ be the $L^2$-norm of $f \in C(G)$. The norm

$$\|f\| = \max(\|f\|_2, |f(e)|)$$

is a normed algebra norm on $C(G)$. The completion $A$ of $C(G)$ in this norm is not Wedderburnian. The Feldman example can be identified with the completion of the socle of $C(G)$, for $G$ the reals modulo one, in the norm $\|f\|$. Other examples arising from algebras of operators on Hilbert space are given. In particular, the completion of the trace class [9, p. 37] of Schatten in an appropriate norm is not Wedderburnian.

2. Preliminary theory

We adopt the following notation. Let $B$ be a Banach algebra in the norm $\|x\|$ and $E$ be a Banach space in the norm $\|\xi\|_E$. Let $T$ be a linear mapping of $B$ into $E$ satisfying

$$\|T(xy)\|_E \leq \|x\| \|y\|$$

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for all \( x, y \) in \( B \). We let \( A \) be the set of all elements of the form \( x + \xi \), where \( x \in B \) and \( \xi \in E \), made into an algebra under the rules \((x + \xi) + (y + \eta) = (x + y) + (\xi + \eta)\), \( a(x + \xi) = (ax + a\xi) \), and \((x + \xi)(y + \eta) = xy\) for all \( x, y \in B \), \( \xi, \eta \in E \), and scalars \( a \).

We define a norm on \( A \) by

\[
|||x + \xi||| = \max(||x||, ||\xi - T(x)||_E).
\]

In view of (1) we see that \(|||x + \xi|||\) is a normed algebra norm on \( A \).

2.1. **Lemma.** \( A \) is a Banach algebra in the norm \(|||x + \xi|||\).

**Proof.** Let \( \{x_n + \xi_n\} \) be a Cauchy sequence in \( A \). Then \( \{x_n\} \) is a Cauchy sequence in \( B \) and \( \{T(x_n) - \xi_n\} \) is a Cauchy sequence in \( E \). Hence there exists \( y \in B \) and \( \eta \in E \) where \( ||x_n - y|| \to 0 \) and \( ||T(x_n) - \xi_n - \eta||_E \to 0 \). One readily checks that, in \( A \), the sequence \( x_n + \xi_n \) has \( y + [T(y) - \eta] \) as its limit.

We denote the radical of \( A \) by \( R \).

2.2. **Lemma.** If \( B \) is semisimple then \( R = E \).

**Proof.** Clearly \( E \subset R \). Note that \( B \) is a two-sided ideal in \( A \). Therefore, \( R \cap B \) is the radical of \( B \) so that \( R \cap B = (0) \). Let \( x + \xi \in R \) where \( x \in B \) and \( \xi \in E \). Then \( x \in R \cap B \) so that \( x = 0 \).

2.3. **Lemma.** Suppose that \( T \) is discontinuous on a linear subspace \( W \) of \( B \). If \( E \) is finite dimensional then the closure of \( W \) in \( A \) must contain a nonzero element of \( E \).

**Proof.** There exists a sequence \( \{x_n\} \) in \( W \) where \( ||x_n|| \to 0 \) and \( ||T(x_n)||_E = 1 \) for each \( n = 1, 2, \ldots \). As \( E \) is finite dimensional, there is a subsequence \( \{y_n\} \) of \( \{x_n\} \) and some \( \xi \neq 0 \) in \( E \) such that \( ||\xi - T(y_n)||_E \to 0 \). Then, since

\[
|||y_n + \xi||| = \max(||y_n||, ||\xi - T(y_n)||_E),
\]

we see that \( -\xi \) is in the closure of \( W \) in \( A \).

2.4. **Theorem.** Suppose that \( B \) is semisimple and that \( E \) is finite dimensional.

Suppose that \( W \) is a two-sided ideal in \( B \) and that \( T \) is discontinuous on \( W^2 \). Then the completion \( V \) of \( W \) in the norm

\[
|||x||| = \max(||x||, ||T(x)||_E)
\]

is a Banach algebra which is not Wedderburnian.

**Proof.** By Lemma 2.1, \( V \) is just the closure of \( W \) in the Banach algebra \( A \).

Also \( V \) is a two-sided ideal in \( A \) so that, by Lemma 2.2, the radical \( S \) of \( V \) is \( V \cap E \).

Suppose that \( V = K \oplus S \) where \( K \) is a subalgebra of \( V \). Since \( sv = vs = 0 \) for all \( s \in S \) and \( v \in V \), we have \( K \supseteq V^2 \supseteq W^2 \). Hence, by Lemma 2.3, the closure of \( K \) in \( V \) must contain a nonzero element of \( S \). Consequently \( V \) is not Wedderburnian.

3. **Examples from harmonic analysis**

Let \( G \) be an infinite compact topological group with identity \( e \) and normalized Haar measure \( m(E) \). We consider \( C(G) \) in the sup norm and \( L^2(G) \) in
the $L^2$-norm $\|f\|_2$ as Banach algebras with convolution $f * g$ as the multiplication. If $f$ and $g$ are in $L^2(G)$ then $f * g$ lies in $C(G)$ by [6, p. 295]. Therefore, the socle $\mathcal{S}$ of $L^2(G)$ lies in $C(G)$, and $\mathcal{S}$ is also the socle of $C(G)$. As $C(G)$ is a dual algebra [7, Theorem 15], $\mathcal{S}$ is dense in $C(G)$ as well as $L^2(G)$.

We use the standard description of $\mathcal{S}$ provided by the Peter-Weyl theorem. Let $\Lambda$ be the set of equivalence classes of finite-dimensional irreducible representations of $G$. For each $\alpha \in \Lambda$ we select an irreducible unitary representation $R^\alpha$ in the class $\alpha$. Suppose $R^\alpha(t)$ is the $n_\alpha$ by $n_\alpha$ matrix $(D^\alpha_{ij}(t))$. Then $\mathcal{S}$ consists of all linear combinations of the functions $D^\alpha_{ij}$, $\alpha \in \Lambda$, $i, j = 1, \ldots, n_\alpha$. The functions $n_\alpha^{1/2}D^\alpha_{ij}$ form an orthonormal basis for $L^2(G)$. Also $D^\alpha_{ij} * D^\beta_{rs} = 0$ if $\alpha \neq \beta$ and

$$n_\alpha D^\alpha_{ij} * n_\alpha D^\alpha_{rs} = n_\alpha \delta_{jr} D^\alpha_{is},$$

where $\delta_{jr}$ is the Kronecker delta.

Let $D$ be the set of all linear combinations of the “diagonal” entries $D^\alpha_{ii}$, $\alpha \in \Lambda$, $i = 1, \ldots, n_\alpha$. The convolution of two different diagonal entries is zero and each $D^\alpha_{ii}$ is a scalar multiple of an idempotent. Therefore, $D$ is a commutative subalgebra of $C(G)$.

3.1. **Lemma.** Consider $D$ as a normed algebra in the $L^2$-norm $\|f\|_2$. Then the linear functional $f \to f(e)$ is discontinuous on $D$.

**Proof.** Let $f = \sum_{k=1}^r k^{-1} D^\alpha_{ik} D^\alpha_{ki}$ be in $D$ where the $D^\alpha_{ik} D^\alpha_{ki}$ are different diagonal entries. Then $f(e) = \sum_{k=1}^r k^{-1}$ and $\|f\|_2 = \sum_{k=1}^r k^{-2} n_\alpha^{1/2}$.

3.2. **Lemma.** The closure of $D$ in either $C(G)$ or $L^2(G)$ is semisimple.

**Proof.** Note that $v = n_\alpha^{1/2} D^\alpha_{ii}$ is an idempotent generator of a minimal one-sided ideal of $C(G)$ or $L^2(G)$. Let $W$ be the closure of $D$ and $z$ be in the radical of $W$. We have, as $W$ is commutative, that $vz = zv = vzv$ is a scalar multiple of the idempotent $v$ and is in the radical of $W$. Hence $D^\alpha_{ii} * z = 0$. It follows from (2) that $D^\alpha_{ii} * z = 0$ for all $\alpha \in \Lambda$ and $r, s = 1, \ldots, n_\alpha$. Hence $\mathcal{S} * z = z * \mathcal{S} = (0)$. Since $\mathcal{S}$ is dense in $C(G)$ and $L^2(G)$ and these are semisimple, we see that $z = 0$.

3.3. **Notation.** The functional $f \to f(e)$ which is defined naturally on $C(G)$ can be extended to a linear functional $\phi(f)$ defined on $L^2(G)$ by [10, Theorem 1.71-A, p. 40].

3.4. **Lemma.** For any $f, g \in L^2(G)$ we have

$$|\phi(f * g)| \leq \|f\|_2 \|g\|_2.$$

**Proof.** As noted earlier, $f * g \in C(G)$. Therefore,

$$|\phi(f * g)| = |f * g(e)| \leq \|f\|_2 \|g\|_2$$

by Schwarz’s inequality.
3.5. **Theorem.** Let $K$ be either the closure of $\mathfrak{D}$ in $L^2(G)$ or any two-sided ideal of $L^2(G)$ containing $\mathfrak{D}$. Then the completion of $K$ in the norm

$$|||f||| = \max[||f||_2, |\phi(f)|]$$

is a Banach algebra which is not Wedderburnian.

**Proof.** Clearly $\mathfrak{D}^2 = \mathfrak{D}$. Theorem 3.5 follows from Lemmas 3.1, 3.2, and 3.4 together with Theorem 2.4. The special case $K = C(G)$ was mentioned in §1.

For the case $K = \mathfrak{G}$ we have a specific result.

3.6. **Corollary.** Let $f$ be a typical element of $\mathfrak{G}$ where

$$f = \sum_{k=1}^{r} a_k D_{ij}^{\alpha_k}.$$

Here each $a_k$ is a scalar and no two $D_{ij}^{\alpha}$ agree in all of $\alpha$, $i$, and $j$. Then the completion of $\mathfrak{G}$ in the norm

$$|||f||| = \max \left\{ \left( \sum_{k=1}^{r} |a_k|^2 / n_{\alpha_k} \right)^{1/2}, \left( \sum_{k=1}^{r} a_k \delta_{i,j,k} \right) \right\}$$

is not Wedderburnian.

**Proof.** We use Theorem 3.5 together with $D_{ij}^{\alpha}(e) = \delta_{ij}$.

If $G$ is abelian each $m_{\alpha} = 1$ and each $\delta_{i,j} = 1$. Here $\mathfrak{G}$ is the set of linear combinations of the continuous characters of $G$.

3.7. **Corollary.** Let $G$ be an abelian compact group whose character group $\hat{G}$ is denumerably infinite: $\hat{G} = \{\gamma_1, \gamma_2, \ldots\}$. Then the completion of $\mathfrak{G}$ in the norm

$$|||r||| = \max \left\{ \left( \sum_{k=1}^{r} |a_k|^2 \right)^{1/2}, \sum_{k=1}^{r} a_k \right\}$$

is not Wedderburnian.

For $G$ the reals modulo one we have, except for a difference in notation, the Feldman example [4].

4. **Examples from operator theory**

Let $B(H)$ be the algebra of all bounded linear operators on a separable infinite-dimensional Hilbert space $H$. Let $\{\phi_n\}$ be an orthonormal basis for $H$. As in Schatten's book [9] (see also [3, Chapter 1]) we consider the Schmidt class $B_2$ and the trace-class $B_1$ of operators on $H$. $B_2$ is the set of all $T \in B(H)$ for which $\sum_j ||T(\phi_j)||^2 < \infty$. This sum is finite and the same if $\{\phi_n\}$ is replaced by another orthononormal basis $\{\psi_n\}$. As shown in [9], $B_2$ is a Banach $*$-algebra in the norm

$$||T||_2 = \left[ \sum_{j} ||T(\phi_j)||^2 \right]^{1/2}.$$

Also $||T||_2 = ||T^*||_2$ for all $T \in B_2$. 
Let $|T|$ be the unique positive square root of $T^*T$. The trace-class $B_1$ is the set of all $T \in B(H)$ for which $\sum_j (|T|(\phi_j), \phi_j) < \infty$. Again this sum is finite and the same if $\{\phi_n\}$ is replaced by another orthonormal basis $\{\psi_n\}$. As shown in [9], $B_1$ is a Banach *-algebra under the norm
\[ \|T\|_1 = \sum_j (|T|(\phi_j), \phi_j). \]
Furthermore, $B_1$ is the set of all products $TU$ for $T, U \in B_2$, and the elements of $B_1$ all have a finite trace
\[ \text{tr}(U) = \sum_j (U(\phi_j), \phi_j), \quad U \in B_1, \]
again independent of the choice of the orthonormal basis. By [10, 1.71-A, p. 40], $\text{tr}(U)$ can be extended to be a linear functional $\text{TR}(U)$ on all of $B_2$.

We note that (see [9]) the common socle of $B_1$ and $B_2$ is the set $F(H)$ of all $U \in B(H)$ with finite-dimensional range.

4.1. **Lemma.** For $U, V \in B_2$ we have
\[ |\text{TR}(UV)| \leq \|U\|_2 \|V\|_2. \]
**Proof.** As noted above, $UV \in B_1$. Therefore,
\[ |\text{TR}(UV)| = \left| \sum_j (UV(\phi_j), \phi_j) \right| \leq \sum_j (V(\phi_j), U^*(\phi_j)) \]
\[ \leq \sum_j \|V(\phi_j)\| \|U^*(\phi_j)\| \leq \|V\|_2 \|U^*\|_2 = \|V\|_2 \|U\|_2. \]

4.2. **Lemma.** $\text{tr}(U)$ is discontinuous on $F(H)$ if $F(H)$ is taken in the norm $\|U\|_2$.
**Proof.** For each positive integer $n$ we define $W_n \in F(H)$ as follows. Let $W_n(\phi_j) = \phi_j/j$ for $j = 1, \ldots, n$ and $W_n(\phi_j) = 0$ for $j > n$. Then
\[ \text{tr}(W_n) = \sum_{j=1}^n j^{-1} \quad \text{and} \quad \|W_n\|_2 = \sum_{j=1}^n j^{-2}. \]

4.3. **Theorem.** Let $K$ be any two-sided ideal of $B_2$ which contains $F(H)$. The completion of $K$ in the norm
\[ \|\|V\|| = \max(\|V\|_2, |\text{TR}(V)|) \]
is not Wedderburnian.
**Proof.** Note that $F(H) = [F(H)]^2$. We can then use Lemmas 4.1 and 4.2 to apply Theorem 2.4. The particular case $K = B_1$ was noted in §1.

Consider the specific case $H = l_2$. Any $V \in B(l_2)$ can be described in matrix terms. There corresponds to $V$ an infinite matrix $[v_{rs}]$ so that, for $x = \{x_n\}$ and $y = \{y_n\}$ in $l_2$, $V(x) = y$ if and only if
\[ y_r = \sum_{s=1}^\infty v_{rs}x_s, \quad r = 1, 2, \ldots. \]
For $V$ we have
\[ \|V\|_2 = \left( \sum_j \sum_{ij} |v_{ij}|^2 \right)^{1/2}, \quad \text{tr}(V) = \sum_j v_{jj}. \]

In these terms $B_2$ is the set of all $V \in B(l_2)$ for which $\sum_j \sum_i |v_{ij}|^2 < \infty$, and although there seems to be no simple description of $B_1$ in matrix terms (see [8, p. 107]), $F(l_2)$ is easily described as all $V \in B_2$ for which the column vectors of the matrix $[v_{rs}]$ lie in a finite-dimensional subspace of $l_2$.

4.4. Corollary. The completion of $F(l_2)$ in the normed algebra norm
\[ \|\|V\|\| = \max \left\{ \left[ \sum_j \sum_i |v_{ij}|^2 \right]^{1/2}, \left[ \sum_j v_{jj} \right] \right\} \]

is not Wedderburnian.

**Added in proof**


**References**