STRONGLY EXTREME POINTS
AND THE RADON-NIKODÝM PROPERTY

ZHIBAO HU

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Abstract. We prove that if $K$ is a bounded and convex subset of a Banach space $X$ and $x$ is a point in $K$, then $x$ is a strongly extreme point of $K$ if and only if $x$ is a strongly extreme point of $\overline{K}$ which is the weak* closure of $K$ in $X^{**}$. We also prove that a Banach space $X$ has the Radon-Nikodym property if and only if for any equivalent norm on $X$, the unit ball has a strongly extreme point.

Suppose $K$ is a subset of a Banach space $X$ and $x \in K$. The element $x$ is called an extreme point of $K$ if $x \notin \text{co}(K \setminus \{x\})$, where $\text{co}(K \setminus \{x\})$ is the convex hull of the set $K \setminus \{x\}$. Various kinds of extreme points have been introduced and studied, among them are denting points and strongly extreme points. Denting points can be defined in terms of slices of $K$ which are of the form

$$S(x^*, K, \delta) = \{x \in K : x^*(x) > \sup x^*(K) - \delta\},$$

where $\delta$ is a positive number and $x^*$ is an element in $X^*$, the dual of $X$. The element $x$ is called a denting point of $K$ if the family of all slices of $K$ containing $x$ is a neighborhood base of $x$ with respect to the relative norm topology on $K$. It is called a strongly extreme point of $K$ if for any $\epsilon > 0$ there is a $\delta > 0$ such that for any $y$ in $X$ the conditions $d(x + y, K) < \delta$ and $d(x - y, K) < \delta$ imply that $\|y\| < \epsilon$, where $d(x, K)$ is the distance between $x$ and $K$. We use $\text{ext} K$ (resp. $\text{str-ext} K$, $\text{dent} K$) to denote the set of the extreme (resp. strongly extreme, denting) points of $K$. It is obvious that if $x \in \text{dent} K$, then $x \in \text{str-ext} K$. In addition, it is easy to see that if $K$ is convex and $x \in \text{str-ext} K$, then $x \in \text{ext} K$. Let $\overline{K}^*$ be the weak* closure of $K$ in $X^{**}$. An extreme point of $K$ may not be an extreme point of $\overline{K}^*$, even if $K$ is the unit ball of $X$ [5]. On the other hand, we will show that if $K$ is bounded and convex and $x \in K$, then $x \in \text{str-ext} K$ if and only if $x \in \text{str-ext} \overline{K}^*$ (see Theorem 3).

Two important properties of Banach spaces, namely, the Radon-Nikodym property (RNP) and the Krein-Milman property (KMP), can be defined in terms of denting points and extreme points respectively. The Banach space $X$ is said
to have the RNP (resp. KMP) if every nonempty bounded closed convex set \( K \) in \( X \) has a denting (resp. extreme) point [1]. It is unknown whether the RNP and the KMP are equivalent. However, using a result of Huff and Morris [3], it can be proved that \( X \) has the RNP if and only if every nonempty bounded closed convex set \( K \) in \( X \) has an extreme point of \( K^* \) [1, Corollary 3.76; 4, Remarks, p. 174]. Morris [5] proved that every separable Banach space that contains an isomorphic copy of \( c_0 \) admits an equivalent strictly convex norm \( \| \| \) such that the unit ball \( B(x, 1) \) of \( X \) has no extreme points of the unit ball \( B(x^*, 1) \) of \( X^* \). On the other hand, it is known that \( X \) has the RNP if and only if for any equivalent norm on \( X \) the respective unit ball \( B_x \) has a denting point (see, e.g., [1, p. 30]). Thus, as observed by Morris [5], if \( X \) has the RNP, then for any equivalent norm on \( X \) the respective unit ball \( B_x \) has an extreme point of \( B_{x^*} \). Morris conjectured [5] that the converse is also true. Though we are not able to prove the conjecture in this paper, we will show that \( X \) has the RNP, if and only if for any equivalent norm on \( X \) the respective unit ball \( B_x \) has a strongly extreme point (see Corollary 6).

For our discussion, we will need several equivalent formulations of strongly extreme points listed in Lemma 1. We omit the proof of Lemma 1 because it is straightforward.

**Lemma 1.** Suppose \( K \) is a subset of a Banach space \( X \) and \( x \in K \). The following are equivalent:

1. \( x \in \text{str-ext } K \).
2. For any sequence \( \{x_n\} \) in \( X \), if \( \lim d(x + x_n, K) = 0 \), then \( \lim x_n = 0 \).
3. For any sequences \( \{x_n\} \) and \( \{y_n\} \) in \( K \), if \( \lim (x_n + y_n)/2 = x \), then \( \lim x_n = \lim y_n = x \).
4. For any \( \varepsilon > 0 \) there is a \( \delta > 0 \), such that for any two vectors \( x_1 \) and \( x_2 \) in \( K \), if \( \| (x_1 + x_2)/2 - x \| < \delta \) then \( \| x_1 - x_2 \| < \varepsilon \).

**Lemma 2.** Suppose \( K \) is a subset of a Banach space \( X \). Let \( Sy(X, K) \) be the convex hull of \( \{(x, 1), (-x, -1) : x \in K\} \) and let \( Sy^*(X, K) \) be the weak* closure of \( Sy(X, K) \) in the bidual of the direct sum \( X \oplus \mathbb{R} \), where \( \mathbb{R} \) is the set of real numbers.

1. The set \( Sy(X, K) \) is symmetric.
2. If \( K \) is bounded, then \( Sy(X, K) \) is bounded and \( Sy^*(X, K) = Sy(X^*, \text{co}^*K) \), where \( \text{co}^*K \) is the weak* closure of \( \text{co} K \) in \( X^* \).
3. If \( K \) is bounded and convex, then \( \text{str-ext } Sy(X, K) = \{(x, 1), (-x, -1) : x \in \text{str-ext } K\} \).

**Proof.** (1) and (2) are obvious. Without loss of generality, we assume the norm on \( X \oplus \mathbb{R} \) is defined by \( \| (x, r) \| = \max\{\|x\|, |r|\} \) for every \( (x, r) \) in \( X \oplus \mathbb{R} \). Let \( A = \{(x, 1) : x \in K\} \), and let \( B = -A \). Since \( Sy(X, K) = \text{co}(A \cup B) \), we have \( \text{str-ext } Sy(X, K) \subset A \cup B \). Thus \( \text{str-ext } Sy(X, K) \subset \text{str-ext } A \cup \text{str-ext } B \). It is obvious that \( \text{str-ext } A = \{(x, 1) : x \in \text{str-ext } K\} \) and \( \text{str-ext } B = \{(-x, -1) : x \in \text{str-ext } K\} \). Let \( M = \sup\{\|z\| : z \in Sy(X, K)\} \), and let \( x \in K \). Note that \( M \geq 1 \). If \( (x, 1) \notin \text{str-ext } Sy(X, K) \), then there is \( \varepsilon > 0 \) such that for any
\( \epsilon/2 > \delta > 0 \) there are \( u_1 \) and \( u_2 \) in \( Sy(X, K) \) satisfying

\[
\| (u_1 + u_2)/2 - (x, 1) \| < \delta/(6M) \quad \text{and} \quad \| u_1 - u_2 \| > \epsilon.
\]

For \( i = 1 \) or 2, there are \( x_i \) and \( y_i \) in \( K \) and \( t_i \) in \([0, 1]\) such that \( u_i = t_i(x_i, 1) + (1 - t_i)(-y_i, -1) \). It follows that \( 2 - t_1 - t_2 \leq \|(u_1 + u_2)/2 - (x, 1)\| < \delta/(6M) \). Thus

\[
\| u_i - (x_i, 1) \| = (1 - t_i)\|(x_i + y_i, 2)\| < \delta/3.
\]

Hence

\[
\| x_1 - x_2 \| = \| (x_1, 1) - (x_2, 1) \| \geq \| u_1 - u_2 \| - \| u_1 - (x_1, 1) \| - \| u_2 - (x_2, 1) \|
\]

\[
> \epsilon - \delta/3 - \delta/3 > \epsilon/2
\]

and

\[
\| (x_1 + x_2)/2 - x \| = \| [(x_1, 1) + (x_2, 1)]/2 - (x, 1) \|
\]

\[
\leq \| (u_1 + u_2)/2 - (x, 1) \| + \| [u_1 - (x_1, 1)]/2 \| + \| [u_2 - (x_2, 1)]/2 \|
\]

\[
< \delta/(6M) + \delta/6 + \delta/6 < \delta.
\]

Therefore, \( x \notin \text{str-ext } K \). Similarly if \((-x, -1) \notin \text{str-ext } Sy(X, K)\), then \( x \notin \text{str-ext } K \). Hence \( \text{str-ext } Sy(X, K) = \{(x, 1), (-x, -1) : x \in \text{str-ext } K \} \).

Q.E.D.

**Theorem 3.** If \( K \subset X \) is bounded and convex and \( x \in K \), then \( x \in \text{str-ext } K \) if and only if \( x \in \text{str-ext } K^* \).

**Proof.** Since \( K \) is a subset of \( K^* \), if \( x \in \text{str-ext } K^* \) then \( x \in \text{str-ext } K \). Now suppose \( x \in \text{str-ext } K \). By Lemma 2, we have \((x, 1) \in \text{str-ext } Sy(X, K)\) and \( Sy^*(X, K) = Sy(X^{**}, K^*) \), and \((x, 1) \in \text{str-ext } Sy(X^{**}, K^*)\) if and only if \( x \in \text{str-ext } K^* \). Passing to \((x, 1)\) and \( Sy(X, K)\) if necessary, we assume that \( K \) is also symmetric. Assume that \( x \notin \text{str-ext } K^* \). Then there are \( \epsilon > 0 \) and a sequence \( \{x_n^*\} \) in \( X^{**} \) such that \( \|x_n^*\| > \epsilon \) and \( d(x \pm x_n^*, K^*) < 1/n \) for each \( n \geq 1 \). For each \( n \geq 1 \), choose \( x_n^* \in S_{x^*} \) such that \( x_n^* (x_n^*) > \epsilon \), and let \( ||x||_n \) be the Minkowski functional determined by \( K + 1/nB_X \). It is obvious that \( B_{(x^*, ||x||_n)} = K^* + 1/nB_{X^{**}} \). Thus \( \| x \pm x_n^* \|_n < 1 \). By the local reflexivity of Banach spaces \([2]\), for each \( n \geq 1 \) there is a linear operator \( T_n \) from \( \text{span}\{x, x_n^*\} \) to \( X \) such that

\[
\| T_n(x \pm x_n^*) \|_n < 1, \quad T_n(x) = x, \quad \text{and} \quad x_n^*(T_n(x_n^*)) = x_n^*(x^*) .
\]

Let \( x_n = T_n(x_n^*) \). Then \( \| x_n \| \geq x_n^*(x_n) = x_n^*(x^*) > \epsilon \) and \( \| x \pm x_n \|_n < 1 \). So \( x \pm x_n \in K + 1/nB_X \), that is, we have \( d(x \pm x_n, K) \leq 1/n \). Therefore, \( x \notin \text{str-ext } K \), which is a contradiction. Hence \( x \in \text{str-ext } K^* \). Q.E.D.

Without assuming \( K \) to be bounded, one can prove that if \( x \in \text{str-ext } K \) then \( x \) is an extreme point of \( \text{ext } K^* \) (see \([4, \text{Remarks, p. 174}]\) or Lemma 4).

**Lemma 4.** Suppose \( K \subset X \) is convex and \( x \in K \). Consider the following statements:

1. For any sequences \( \{x_n\} \) and \( \{y_n\} \) in \( K \), if \( \lim_n (x_n + y_n)/2 = x \), then \( \text{weak-lim}_n x_n = \text{weak-lim}_n y_n = x \).
2. For any nets \( \{x_\lambda\} \) and \( \{y_\lambda\} \) in \( K \), if \( \text{weak-lim}_\lambda (x_\lambda + y_\lambda)/2 = x \), then \( \text{weak-lim}_\lambda x_\lambda = \text{weak-lim}_\lambda y_\lambda = x \).
(3) The element \( x \) is an extreme point of \( \overline{K}^* \).

(4) For any bounded sequences \( \{x_n\} \) and \( \{y_n\} \) in \( K \), if \( \lim_n (x_n + y_n)/2 = x \), then weak-lim \( x_n = \text{weak-lim}_n y_n = x \).

(5) For any bounded nets \( \{x_i\} \) and \( \{y_i\} \) in \( K \), if \( \text{weak-lim}_\lambda (x_\lambda + y_\lambda)/2 = x \), then weak-lim \( x_\lambda = \text{weak-lim}_\lambda y_\lambda = x \).

Then (1) and (2) are equivalent and each of them implies (3). Statements (4) and (5) are equivalent and both are implied by (3). Thus if, in addition, the set \( K \) is bounded, then all the above statements are equivalent.

Proof. It is obvious that (2) implies (1). To prove the converse is true, we assume that there exist some nets \( \{x_i\} \) and \( \{y_i\} \) in \( K \) such that weak-lim \( x_i + y_i)/2 = x \), but \( \{x_i\} \) does not converge weakly to \( x \). Passing to subnets if necessary, we may assume that there is \( x^* \) in \( X^* \) such that \( x^*(x_i - x) > 1 \). Since weak-lim \( (x_i + y_i)/2 = x \), we may assume that \( x^*(x - y_i) > 0 \). Let \( A = \text{co}\{x_i\} \) and \( B = \text{co}\{y_i\} \). Then \( \inf x^*(A) \geq \max x^*(B) + 1 \) and there is a sequence \( z_n \) in \( \text{co}\{(x_i + y_i)/2\} \) such that \( \lim_n z_n = x \). Hence there are sequences \( \{x_n\} \) in \( \text{co}\{x_i\} \) and \( \{y_n\} \) in \( \text{co}\{y_i\} \) such that \( (x_n + y_n)/2 = z_n \). Thus \( \lim_n (x_n + y_n)/2 = x \) and \( x^*(x_n - y_n) > 1 \), which imply that either \( \{x_n\} \) or \( \{y_n\} \) does not converge weakly to \( x \). Therefore, (1) implies (2).

The proof of the equivalence of (4) and (5) is similar.

Assume that \( x \) is not an extreme point of \( \overline{K}^* \). Then there are \( x^{**} \) and \( y^{**} \) in \( \overline{K}^* \) such that \( x^{**} \neq x \neq y^{**} \) and \( x = (x^{**} + y^{**})/2 \). Choose \( x^* \) in \( X^* \) such that \( (x^{**} - x)(x^*) > 1 \). Then \( (x - y^{**})(x^*) > 1 \). There exist nets \( \{x_i\} \) and \( \{y_i\} \) in \( K \) such that weak-lim \( x_i = x^{**} \) and weak-lim \( y_i = y^{**} \). Thus weak-lim \( x_i + y_i)/2 = x \), but \( \{x_i\} \) does not converge weakly to \( x \). Hence (2) implies (3).

Finally, to show (3) implies (5), we assume that \( x \) is an extreme point of \( \overline{K}^* \). Suppose \( \{x_i\} \) and \( \{y_i\} \) are two bounded nets in \( K \) with weak-lim \( x_i + y_i)/2 = x \). Then \( \{x_i\} \) has a weak* cluster point. Let \( x^{**} \) be a weak* cluster point of \( \{x_i\} \). Then \( x^{**} \in \overline{K}^* \) and there is a subnet \( \{x_{i(\alpha)}\} \) of \( \{x_i\} \) such that weak-lim \( x_{i(\alpha)} = x^{**} \). Since weak-lim \( x_i + y_i)/2 = x \), the weak* limit of \( \{y_{i(\alpha)}\} \) exists, say, weak-lim \( y_{i(\alpha)} = y^{**} \). Then \( y^{**} \in \overline{K}^* \) and \( x = (x^{**} + y^{**})/2 \). Since \( x \) is an extreme point of \( \overline{K}^* \), we can conclude that \( x^{**} = x \). Hence weak-lim \( x_{i(\alpha)} = x \). Q.E.D.

Theorem 5. Suppose \( K_1, K_2 \subset X \) are closed and convex, and one of them is bounded and \( x \in X \). Let \( K \) be the weak* closure of \( K_1 + K_2 \) in \( X^{**} \). If \( x \) is an extreme point of the weak* closure of \( K \) in \( X^{(4)} \), the fourth dual of \( X \), then \( x \) is in \( K_1 + K_2 \). In particular, if \( x \) is a strongly extreme point of the norm closure of \( K_1 + K_2 \), then \( x \) is in \( K_1 + K_2 \).

Proof. It is obvious that the weak* closure of \( K_1 + K_2 \) is \( \overline{K}_1^* + \overline{K}_2^* \). Thus there are \( u_1 \in \overline{K}_1^* \) and \( u_2 \in \overline{K}_2^* \) such that \( x = u_1 + u_2 \). We can choose sequences \( \{x_1(n)\} \) in \( K_1 \) and \( \{x_2(n)\} \) in \( K_2 \) such that \( \lim_n x_1(n) + x_2(n) = x \). Let \( y_1(n) = x_1(n) + u_2 \) and \( y_2(n) = x_2(n) + u_1 \). Then \( \{y_1(n)\} \) and \( \{y_2(n)\} \) are bounded sequences in \( \overline{K}_1^* + \overline{K}_2^* \). Since \( \lim_n [y_1(n) + y_2(n)]/2 = x \), we have weak-lim \( y_1(n) = \text{weak-lim}_n y_2(n) = x \). Thus by Lemma 4 the sequence \( \{x_i(n)\} \) is weakly convergent for \( i = 1 \) and \( 2 \). It follows that \( u_i \in K_i \). Therefore, \( x \) is in \( K_1 + K_2 \). Now suppose \( x \) is a strongly extreme point of...
the norm closure of \( K_1 + K_2 \). By Theorem 3, the element \( x \) is also a strongly extreme point of \( K \). Thus \( x \) is an extreme point of the weak* closure of \( K \) in \( X^{(4)} \). Therefore, \( x \) is in \( K_1 + K_2 \). Q.E.D.

**Corollary 6.** Let \( X \) be a Banach space. The following are equivalent:

1. The space \( X \) has the RNP.
2. For any equivalent norm \( \| \) on \( X \), the unit ball \( B_{(X, \| \|)} \) has a strongly extreme point.
3. For any equivalent norm \( \| \) on \( X \), the unit ball \( B_{(X, \| \|)} \) has an extreme point of \( B_{(X^{(4)}, \| \|)} \).

**Proof.** It is obvious that (1) implies (2). By Theorem 3, every strongly extreme point of \( B_X \) is an extreme point of \( B_{X^{(4)}} \). Thus (2) implies (3). So it remains to show that (3) implies (1). Let \( A \subset X \) be nonempty, bounded, and weakly closed. Let \( K = \overline{co}(A \cup -A) \), and let \( \| \) be the Minkowski functional determined by \( K + B_X \) where \( B_X \) is the unit ball of \( X \) with respect to the original norm. Then \( \| \) is an equivalent norm on \( X \) such that the unit ball \( B_{(X, \| \|)} \) is the norm closure of \( K + B_X \). Let \( x \) be an element in \( X \) such that \( x \) is an extreme point of \( B_{(X^{(4)}, \| \|)} \) of \( X^{(4)} \). By Theorem 5, there are \( y \in K \) and \( z \in B_X \) such that \( x = y + z \). It is obvious that \( x \) is an extreme point of \( B_{(X^{(4)}, \| \|)} \) and \( B_{(X^{(4)}, \| \|)} = \overline{K} + B_X \). Thus \( y \) is an extreme point of \( \overline{K} \). By the Krein-Milman Theorem, the set \( \text{ext} \overline{K} \) is contained in the weak* closure of \( A \cup -A \). Since \( A \) is weakly closed, we have \( y \in A \) or \( y \in -A \). In any case the set \( A \) has an extreme point. Therefore, \( X \) has the RNP [1, Corollary 3.76]. Q.E.D.

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**References**


**Department of Mathematics and Statistics, Miami University, Oxford, Ohio 45056**

**E-mail address**: zhu@miavx1.acs.muohio.edu