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EQUATIONS $au_n^l = bu_m^k$ SATISFIED BY MEMBERS OF RECURRENCE SEQUENCES

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ABSTRACT. Let $\{u_n\}_{n\in\mathbb{Z}}$ be a linear recurrence sequence. Given $a \neq 0$, $b \neq 0$, and natural $k \neq l$, we study equations as indicated in the title in unknowns n, m. It turns out that under natural conditions on the sequence $\{u_n\}$, there are only finitely many solutions.

1. INTRODUCTION

Let a and b be nonzero complex numbers and l, k natural numbers with $l \neq k$. We will study equations

$$au_n^l = bu_m^k \tag{1.1}$$

in unknowns $(n, m) \in \mathbb{Z}^2$. Here we suppose that $\{u_n\}_{n\in\mathbb{Z}}$ is a linear recurrence sequence, i.e., a sequence satisfying a relation

$$u_{n+t} = \nu_{t-1}u_{n+t-1} + \cdots + \nu_0 u_n \qquad (n \in \mathbb{Z}) \tag{1.2}$$

where $t > 0$ and $\nu_{t-1}, \ldots, \nu_0$ are given, with $\nu_0 \neq 0$. We will suppose that the sequence $\{u_n\}_{n\in\mathbb{Z}}$ is not identically zero.

Consider the companion polynomial

$$P(z) = z^t - \nu_{t-1}z^{t-1} - \cdots - \nu_0 = \prod_{i=1}^{r}(z-\alpha_i)^{\tau_i}, \tag{1.3}$$

say, with distinct roots $\alpha_1, \ldots, \alpha_r$ and multiplicities $\tau_i > 0$. It is well known that there exist polynomials $f_i(x)$ ($i = 1, \ldots, r$) such that $\deg f_i < \tau_i$ and

$$u_n = \sum_{i=1}^{r} f_i(n)\alpha_i^n \qquad (n \in \mathbb{Z}). \tag{1.4}$$

Vice versa, any sequence $\{u_n\}_{n\in\mathbb{Z}}$ as in (1.4) satisfies the recurrence relation (1.1) with coefficients ν_i defined by (1.3). $\{u_n\}$ is called *nondegenerate* if the quotients α_i/α_j for $i \neq j$ are not roots of 1. The correspondence between (1.4) and (1.2) implies that (1.1) is a special instance of an equation

$$v_n = w_m, \tag{1.5}$$

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where $\{v_n\}_{n\in\mathbb{Z}}$ and $\{w_m\}_{m\in\mathbb{Z}}$ are recurrence sequences. The classical theorem of Skolem-Mahler-Lech says that if $\{v_n\}$ is nondegenerate and nonperiodic, then an equation $v_n = c$ has only finitely many solutions $n \in \mathbb{Z}$. In the case when the v_n are algebraic, Schlickewei [5] and van der Poorten and Schlickewei [4] have given a quantitative version of this result. Evertse [1] proved that the equation $v_n = v_m$ admits only finitely many solutions $n \neq m$ with $n, m \geq 0$. Laurent [2] obtained qualitative results about equations $v_n = \lambda v_m$ and more generally about equations (1.5). A complete qualitative characterization of the set of solutions of such equations is given in Schlickewei and Schmidt [7]. For quantitative results about (1.5) we refer the reader to [8]. However, in the context of (1.1), much more can be shown than what follows from the general results in [2, 7, 8]. The results of [2, 7] say that the set of solutions of an equation (1.5) may consist of the union of a finite set and either a finite set of linear 1-parameter families or a finite set of exponential 1-parameter families. Such 1-parameter families do not occur in the set of solutions of (1.1). In fact we will see that apart from trivial exceptions, (1.1) admits only finitely many solutions.

We call t the *order* of the recurrence sequence $\{u_n\}$ if $t > 0$ is least such that a relation like (1.2) holds. For this value of t the relation (1.2) is unique, and the number r of roots α_i of the companion polynomial $P(z)$ is called the *rank* of the recurrence sequence.

We define the sequence $\{u_n\}_{n\in\mathbb{Z}}$ to be *exceptional with respect to* $a, b, l,$ *and* k or briefly *exceptional* if

$$u_n = f(n)\alpha^n,$$

i.e., if $\{u_n\}$ has rank 1 and if moreover one of the following two alternatives is satisfied.

(i) α is a root of unity.

(ii) $f(n)$ is a constant c and there exists an integer $v \equiv 0 \mod \gcd(l, k)$ with $ac^l = bc^k\alpha^v$.

If $\{u_n\}$ is exceptional of type (i), then equation (1.1) reduces essentially to equations

$$a_1 f^l(n) = b_1 f^k(m),$$

which clearly may have infinitely many solutions. A trivial example is $a = b = \alpha = 1$, $f(n) = n$, in which case (1.1) becomes $n^l = m^k$ with the 1-parameter family of solutions $n = q^k$, $m = q^l$ $(q \in \mathbb{Z},\ q \geq 0)$.

If $\{u_n\}$ is exceptional of type (ii), then writing v in (ii) as $v = -cl + dk$ with integers c and d and defining the natural numbers l_1, k_1 by $l \cdot l_1 = k \cdot k_1 = \operatorname{lcm}\{l, k\}$, it is seen that (1.1) has the 1-parameter family of solutions

$$n = l_1 w + c, \quad m = k_1 w + d \qquad (w \in \mathbb{Z}).$$

Theorem 1. *Suppose that the linear recurrence sequence* $\{u_n\}_{n\in\mathbb{Z}}$ *is nondegenerate and not exceptional. Then equation* (1.1) *has only finitely many solutions* $(n, m) \in \mathbb{Z}^2$.

Theorem 1 deals with arbitrary complex recurrence sequences. We can give a quantitative version of our result if we suppose that the elements u_n are algebraic numbers. Let $\mathscr{K}$ be a number field. We say that $\{u_n\}_{n\in\mathbb{Z}}$ is *defined* *over* $\mathscr{K}$ if the zeros α_i of the companion polynomial (1.3) as well as the coefficients of the polynomials f_i in (1.4) are contained in $\mathscr{K}$. We denote by S the set of archimedean absolute values of $\mathscr{K}$ together with those nonarchimedean absolute values $|\ |_v$ of $\mathscr{K}$ having $|\alpha_i|_v \neq 1$ for some i ($1 \leq i \leq r$). S is a finite set and we write $s = |S|$ for its cardinality.

Theorem 2. *Let* $\mathscr{K}$ *be a number field of degree* d. *Let* $a, b \in \mathscr{K}^*$ *be given. Suppose that* $l < k$. *Finally, let* $\{u_n\}_{n\in\mathbb{Z}}$ *be a linear recurrence sequence of order* t *and rank* $r > 1$, *which is defined over* $\mathscr{K}$. *Then the number of solutions* $(n, m) \in \mathbb{Z}^2$ *of equation* (1.1) *does not exceed*

$$2^{s^7 2^{44d!(4^{t+k})!}}. \tag{1.6}$$

The significant feature in the bound (1.6) is its uniformity. The sequences $\{u_n\}$ with (1.2) make up a vector space of dimension t, but the bound (1.6) holds uniformly for the nonzero sequences $\{u_n\}$ with (1.2). Moreover, the bound does not depend upon the recurrence coefficients $\nu_{t-1}, \dots, \nu_0$, or more precisely it depends upon these coefficients only via the parameter s. Notice that s implicitly involves the degree d of $\mathscr{K}$ and (by the nonarchimedean valuations) the number of prime ideal divisors needed in the decomposition of the fractional ideals (α_i) ($1 \leq i \leq r$) in $\mathscr{K}$.

We remark that it would have been possible to derive a quantitative result also in the rank 1 case; however, then we do obtain a bound that is not uniform but involves the coefficients of the polynomials f_i in (1.4). Implicitly, however, this question is treated in the authors' forthcoming paper [8].

2. Polynomial-exponential equations

Consider equations

$$\sum_{i=1}^{h} f_i(\underline{\underline{x}})\underline{\underline{\alpha}}_i^{\underline{\underline{x}}} = 0 \tag{2.1}$$

in variables $\underline{\underline{x}} = (x_1, \dots, x_N) \in \mathbb{Z}^N$, where the f_i are polynomials and where $\underline{\underline{\alpha}}_i^{\underline{\underline{x}}} = \alpha_{i1}^{x_1} \cdots \alpha_{iN}^{x_N}$ with nonzero complex numbers α_{ij} ($1 \leq i \leq h$, $1 \leq j \leq N$). When $\mathscr{P}$ is a partition of $\{1, \dots, h\}$ and π a subset of $\{1, \dots, h\}$, write $\pi \in \mathscr{P}$ if π is among the subsets belonging to $\mathscr{P}$. Consider the system of equations

$$\sum_{i\in\pi} f_i(\underline{\underline{x}})\underline{\underline{\alpha}}_i^{\underline{\underline{x}}} = 0 \qquad (\pi \in \mathscr{P}). \tag{2.1$\mathscr{P}$}$$

Denote by $\mathfrak{S}(\mathscr{P})$ the set of solutions $\underline{\underline{x}} \in \mathbb{Z}^N$ of (2.1$\mathscr{P}$) which do not satisfy (2.1$\mathscr{Q}$) for any proper refinement $\mathscr{Q}$ of $\mathscr{P}$. It is clear that for every solution $\underline{\underline{x}} \in \mathbb{Z}^N$ of (2.1) there exists at least one $\mathscr{P}$ such that $\underline{\underline{x}} \in \mathfrak{S}(\mathscr{P})$. Therefore if we want to find an upper bound for the cardinality of the set of solutions of (2.1), it suffices to find upper bounds for the cardinality of the sets $\mathfrak{S}(\mathscr{P})$. Write $i \overset{\mathscr{P}}{\sim} j$ if $i, j \in \{1, \dots, h\}$ belong to the same subset of $\mathscr{P}$. Denote by $H(\mathscr{P})$ the subgroup of $\mathbb{Z}^N$ consisting of points $\underline{\underline{x}}$ having

$$\underline{\underline{\alpha}}_i^{\underline{\underline{x}}} = \underline{\underline{\alpha}}_j^{\underline{\underline{x}}} \quad \text{for every } i, j \text{ with } i \overset{\mathscr{P}}{\sim} j.$$

The following result is due to Laurent [2].

Theorem A. *Suppose that* $H(\mathscr{P}) = \{\underline{0}\}$. *Then* $\mathfrak{S}(\mathscr{P})$ *is finite.*

In fact this is a special case of Théorème 1 of [2].

Theorem A will be the basic tool in the proof of Theorem 1. To derive Theorem 2, we need a quantitative version of Theorem A. Such a version was derived recently by the authors [6]. We suppose that the coefficients of the polynomials f_i in (2.1) as well as the α_{ij} lie in our number field $\mathscr{K}$. We denote by S' the set of archimedean absolute values of $\mathscr{K}$ together with those non-archimedean absolute values having $|\alpha_{ij}|_v \neq 1$ for some pair i, j with $1 \leq i \leq h$, $1 \leq j \leq N$, and we write $s' = |S'|$ for its cardinality. Furthermore we define $D = \binom{N+\delta}{N}$, where δ is the maximum total degree of the polynomials f_i. Under these assumptions the result of [6] is as follows.

Theorem B. *Suppose* $H(\mathscr{P}) = \{\underline{0}\}$. *Then* $\mathfrak{S}(\mathscr{P})$ *has cardinality*

$$|\mathfrak{S}(\mathscr{P})| < 2^{20N^4 + Ns'^7 2^{43d!(Dh)!}}. \tag{2.2}$$

We mention that both Theorem A and Theorem B are consequences of the Subspace Theorem in diophantine approximation.

3. The general setting

Let L be the set of l-tuples $\lambda = (i_1, \dots, i_l)$ with $1 \leq i_1 \leq \cdots \leq i_l \leq r$, and let K be the set of k-tuples $\kappa = (j_1, \dots, j_k)$ with $1 \leq j_1 \leq \cdots \leq j_k \leq r$. Set $T = L \cup K$, so that by (1.3), and assuming $l < k$, say, T has cardinality

$$|T| = |L| + |K| = \binom{r+l-1}{l} + \binom{r+k-1}{k} \leq 2^{r+l-1} + 2^{r+k-1} < 2^{t+k}. \tag{3.1}$$

For $\lambda \in L$, set $\varphi_\lambda = \alpha_{i_1} \cdots \alpha_{i_l}$, and for $\kappa \in K$, set $\varphi_\kappa = \alpha_{j_1} \cdots \alpha_{j_k}$.

In view of (1.4), equation (1.1) may be written as

$$\sum_{\lambda \in L} F_\lambda(n) \varphi_\lambda^n + \sum_{\kappa \in K} G_\kappa(m) \varphi_\kappa^m = 0, \tag{3.2}$$

where for $\lambda = (i_1, \dots, i_l) \in L$ the polynomial $F_\lambda(n)$ is an integral multiple of $af_{i_1}(n) \cdots f_{i_l}(n)$, and similarly for $\kappa = (j_1, \dots, j_k) \in K$, $G_\kappa(m)$ is an integral multiple of $bf_{j_1}(m) \cdots f_{j_k}(m)$. We will apply the theorems quoted in §2 to equation (3.2), or, to put things more clearly, to the equation

$$\sum_{\lambda \in L} F_\lambda(n) \cdot \varphi_\lambda^n \cdot 1^m + \sum_{\kappa \in K} G_\kappa(m) \cdot 1^n \cdot \varphi_\kappa^m = 0. \tag{3.3}$$

We will treat separately sequences of rank 1 and sequences of rank $r > 1$.

When $r > 1$, we will study partitions $\mathscr{P}$ of the set T. We will distinguish between partitions $\mathscr{P}$ that contain a singleton, i.e., a one-element subset π, and other partitions. Using Theorem A we will show that for each partition $\mathscr{P}$ of T the set $\mathfrak{S}(\mathscr{P})$ is finite. Using Theorem B, and under the hypotheses of Theorem 2, we will give an upper bound for the cardinality $|\mathfrak{S}(\mathscr{P})|$.

4. Sequences of rank 1

Lemma 1. *Suppose* A, B *lie in* $\mathbb{C}^*$ *and are not roots of* 1. *Suppose that* $F(X)$ *and* $G(X)$ *are polynomials of degree* ≥ 1 *with complex coefficients. Let* M *be the set of pairs* $(n, m) \in \mathbb{Z}^2$ *satisfying*

$$F(n)A^n = G(m)B^m, \tag{4.1}$$

and suppose that M contains infinitely many elements. Then there exist integers $p \neq 0$, $q \neq 0$ such that

$$A^p = B^q. \tag{4.2}$$

Moreover, given a pair (p, q) with (4.2), *there exists an integer e and a finite subset M_0 of $\mathbb{Z}^2$ such that M is contained in the union of M_0 and the set*

$$M_1 = \{(n, m) \in \mathbb{Z}^2 \mid pm - qn = e\}.$$

This is a special case of Lemma 6 of Laurent [2].

We proceed to deduce Theorem 1 for sequences $\{u_n\}_{n\in\mathbb{Z}}$ of rank 1. By (1.4) we may write $u_n = f(n)\alpha^n$, where $f(n)$ is a polynomial and $\alpha \in \mathbb{C}^*$. Suppose first that f is constant. Then, if (1.1) has a solution $(n, m) \in \mathbb{Z}^2$ at all, the sequence $\{u_n\}$ will be exceptional; but this is excluded in Theorem 1. Thus, we may assume that $\deg f \geq 1$. Then (1.1) reads

$$af^l(n)(\alpha^l)^n = bf^k(m)(\alpha^k)^m.$$

Since our sequence is not exceptional, α is not a root of unity. Therefore, we may apply Lemma 1 with $A = \alpha^l$, $B = \alpha^k$, $F(X) = af^l(X)$, $G(X) = bf^k(X)$. Now (4.2) is satisfied with $p = k$ and $q = l$, and by Lemma 1 it suffices to show that (1.1) has only finitely many solutions (n, m) satisfying $km - ln = e$ where e is fixed; but then (1.1) reduces to the polynomial equation

$$F(n) = G\left(\frac{l}{k}\, n + e\right)\alpha^{ke}.$$

However, since $l \neq k$, we have $\deg F \neq \deg G$. We may conclude that there will be only finitely many possibilities for n. This proves Theorem 1 for sequences $\{u_n\}$ of rank 1.

5. Groups $H(\mathscr{P})$

In the remainder of the paper we shall assume that the rank r of our sequence $\{u_n\}_{n\in\mathbb{Z}}$ satisfies

$$r > 1. \tag{5.1}$$

Recall the definition of the set T in §3. Let $\mathscr{P}$ be a partition of T. Given $\tau_1, \tau_2 \in T$, write $\tau_1 \sim \tau_2$ if they belong to the same set of $\mathscr{P}$. In view of (3.3), $H(\mathscr{P})$ in our situation is the subgroup of $\mathbb{Z}^2$ consisting of pairs (p, q) such that

$$\varphi_{\lambda_1}^p = \varphi_{\lambda_2}^p \quad \text{whenever } \lambda_1 \sim \lambda_2, \tag{5.2}$$

$$\varphi_{\kappa_1}^q = \varphi_{\kappa_2}^q \quad \text{whenever } \kappa_1 \sim \kappa_2, \tag{5.3}$$

$$\varphi_{\lambda}^p = \varphi_{\kappa}^q \quad \text{whenever } \lambda \sim \kappa. \tag{5.4}$$

Proposition. *Suppose that $\mathscr{P}$ contains no singleton, i.e., a* 1*-element set. Then*

$$H(\mathscr{P}) = \{(0, 0)\}.$$

Proof. We will suppose that $\mathscr{P}$ contains no singleton, but that $H(\mathscr{P})$ contains a pair $(p, q) \neq (0, 0)$, and we will reach a contradiction.

Suppose that the multiplicative group generated by $\alpha_1, \dots, \alpha_r$ has rank g. Then there exist multiplicatively independent elements $\beta_1, \dots, \beta_g$ in $\mathbb{C}^*$ such that we have

$$\alpha_i = \varepsilon_i \beta_1^{x_{i1}} \cdots \beta_g^{x_{ig}} \qquad (i = 1, \dots, r),$$

where $\varepsilon_1, \dots, \varepsilon_r$ are roots of unity and the $x_{ij} \in \mathbb{Z}$. The exponent vectors $\underline{\underline{x}}_i = (x_{i1}, \dots, x_{ig})$ for $i = 1, \dots, r$ are distinct, since α_i/α_j for $i \neq j$ is not a root of 1. Choose an integer point $\underline{\underline{x}} \in \mathbb{R}^g$ such that the inner products $\langle \underline{\underline{x}}_i, \underline{\underline{x}} \rangle$ for $i = 1, \dots, r$ are distinct integers. (A relation $\langle \underline{\underline{x}}_i, \underline{\underline{x}} \rangle = \langle \underline{\underline{x}}_j, \underline{\underline{x}} \rangle$ ($i \neq j$) defines a subspace of codimension 1 in $\mathbb{R}^g$, and we simply have to take $\underline{\underline{x}}$ outside the union of $\binom{r}{2}$ such subspaces.) Set $v_i = \langle \underline{\underline{x}}_i, \underline{\underline{x}} \rangle$. When $\lambda = (i_1, \dots, i_l) \in L$, put $\sigma(\lambda) = v_{i_1} + \cdots + v_{i_l}$, and for $\kappa = (j_1, \dots, j_k) \in K$, put $\sigma(\kappa) = v_{j_1} + \cdots + v_{j_k}$. Since $\beta_1, \dots, \beta_g$ are multiplicatively independent, (5.2), (5.3), (5.4) respectively yield

$$p\sigma(\lambda_1) = p\sigma(\lambda_2) \quad \text{whenever } \lambda_1 \sim \lambda_2, \tag{5.5}$$

$$q\sigma(\kappa_1) = q\sigma(\kappa_2) \quad \text{whenever } \kappa_1 \sim \kappa_2, \tag{5.6}$$

$$p\sigma(\lambda) = q\sigma(\kappa) \quad \text{whenever } \lambda \sim \kappa. \tag{5.7}$$

What we have done so far is to replace the multiplicative subgroup of $\mathbb{C}^*$ generated by $\alpha_1, \dots, \alpha_r$ by the additive subgroup of $\mathbb{Z}$ generated by $v_1, \dots, v_r$.

We may suppose without loss of generality that

$$v_1 < \cdots < v_r. \tag{5.8}$$

Set $\lambda_{Ma} = (r, \dots, r)$, $\lambda_{ma} = (r-1, r, \dots, r)$, $\lambda_{Mi} = (1, \dots, 1)$, $\lambda_{mi} = (1, \dots, 1, 2)$. Then $\sigma(\lambda_{ma}) < \sigma(\lambda_{Ma})$, and any λ distinct from $\lambda_{ma}, \lambda_{Ma}$ has $\sigma(\lambda) < \sigma(\lambda_{ma})$. Similarly $\sigma(\lambda_{Mi}) < \sigma(\lambda_{mi})$, and any λ distinct from $\lambda_{Mi}, \lambda_{mi}$ has $\sigma(\lambda_{mi}) < \sigma(\lambda)$. (When $r = 2$ and $l = 2$, then $\lambda_{mi} = \lambda_{ma}$, and when $r = 2$ and $l = 1$, then $\lambda_{mi} = \lambda_{Ma}$ and $\lambda_{ma} = \lambda_{Mi}$.) Similarly define $\kappa_{Ma} = (r, \dots, r)$, $\kappa_{ma} = (r-1, r, \dots, r)$, $\kappa_{Mi} = (1, \dots, 1)$, $\kappa_{mi} = (1, \dots, 1, 2)$ (but now with k components rather than l). We have analogous properties as above. We may suppose that $k > l$.

Case 1. $pq = 0$. Say $p = 0$, $q \neq 0$. Since $\mathscr{P}$ contains no singleton, either $\kappa_{Ma} \sim \kappa$ with $\kappa_{Ma} \neq \kappa$, or $\kappa_{Ma} \sim \lambda$ for some λ. These two subcases by (5.6), (5.7) yield respectively $\sigma(\kappa_{Ma}) = \sigma(\kappa)$ and $\sigma(\kappa_{Ma}) = 0$. Since the first case is impossible, $\sigma(\kappa_{Ma}) = 0$. But an analogous argument gives $\sigma(\kappa_{Mi}) = 0$, and we have a contradiction.

Case 2. *p, q are of the same sign.* Without loss of generality, $p > 0$, $q > 0$.

(i) $r = 2$, $l = 1$. Then $k > 1$ and therefore

$$\begin{cases} p\sigma(\lambda_{Mi}) < p\sigma(\lambda_{Ma}), \\ q\sigma(\kappa_{Mi}) < q\sigma(\kappa_{mi}) \leq \cdots \leq q\sigma(\kappa_{ma}) < q\sigma(\kappa_{Ma}). \end{cases} \tag{5.9}$$

Notice that in this case we have only two different points $\lambda \in L$, namely, λ_{Mi} and λ_{Ma}. On the other hand K contains at least the three elements $\kappa_{Mi}, \kappa_{mi}, \kappa_{Ma}$. Moreover, any $\kappa \neq \kappa_{Mi}, \kappa \neq \kappa_{mi}$ has $q\sigma(\kappa_{mi}) < q\sigma(\kappa)$; but then (5.9) implies that $\kappa_{Mi}, \kappa_{mi}, \kappa_{Ma}$ do not satisfy a relation (5.6). On the other hand, as there are only two different values $p\sigma(\lambda)$, again by (5.9) not all three of them can satisfy a relation (5.7). We may conclude that $\mathscr{P}$ contains a singleton $\{\kappa\}$, which contradicts our hypothesis.

(ii) $r = 2$, $l = 2$. Then $k \geq 3$. Now L consists of the elements $\lambda_{Mi}, \lambda_{mi}, \lambda_{Ma}$, whereas K contains at least the four distinct elements κ_{Mi}, $\kappa_{mi}, \kappa_{ma}, \kappa_{Ma}$. We get

$$p\sigma(\lambda_{Mi}) < p\sigma(\lambda_{mi}) < p\sigma(\lambda_{Ma}),$$
$$q\sigma(\kappa_{Mi}) < q\sigma(\kappa_{mi}) < \cdots < q\sigma(\kappa_{ma}) < q\sigma(\kappa_{Ma}),$$

and the same argument as in subcase (i) shows that $\mathscr{P}$ contains a singleton $\{\kappa\}$. Again we reach a contradiction.

(iii) $l \geq 3$. We have

$$p\sigma(\lambda_{Mi}) < p\sigma(\lambda_{mi}) < \cdots < p\sigma(\lambda_{ma}) < p\sigma(\lambda_{Ma}),$$
$$q\sigma(\kappa_{Mi}) < q\sigma(\kappa_{mi}) < \cdots < q\sigma(\kappa_{ma}) < q\sigma(\kappa_{Ma}).$$

By this we mean, e.g., that $p\sigma(\lambda) < p\sigma(\lambda_{ma})$ if $\lambda \neq \lambda_{ma}, \lambda_{Ma}$, and that $p\sigma(\lambda_{mi}) < p\sigma(\lambda)$ if $\lambda \neq \lambda_{Mi}, \lambda_{mi}$. Since $\mathscr{P}$ has no singleton, (5.5), (5.6), (5.7) yield

$$\begin{aligned}
\lambda_{Mi} \sim \kappa_{Mi} &\quad \text{and} \quad p\sigma(\lambda_{Mi}) = q\sigma(\kappa_{Mi}),\\
\lambda_{Ma} \sim \kappa_{Ma} &\quad \text{and} \quad p\sigma(\lambda_{Ma}) = q\sigma(\kappa_{Ma}),\\
\lambda_{mi} \sim \kappa_{mi} &\quad \text{and} \quad p\sigma(\lambda_{mi}) = q\sigma(\kappa_{mi}),\\
\lambda_{ma} \sim \kappa_{ma} &\quad \text{and} \quad p\sigma(\lambda_{ma}) = q\sigma(\kappa_{ma}).
\end{aligned}$$

These four equations in turn yield

$$\begin{aligned}
plv_1 &= qkv_1,\\
plv_r &= qkv_r,\\
p((l-1)v_1 + v_2) &= q((k-1)v_1 + v_2),\\
p((l-1)v_r + v_{r-1}) &= q((k-1)v_r + v_{r-1}).
\end{aligned}$$

Since $v_1 \neq v_r$, the first two give $pl = qk$, and then the third gives $p(v_2 - v_1) = q(v_2 - v_1)$, therefore $p = q$. Since $(p, q) \neq (0, 0)$, we may conclude that $l = k$, which is impossible.

Case 3. *p, q are of opposite sign.* Say $p > 0$, $q < 0$. Again we have the subcases (i) $r = 2$, $l = 1$ and (ii) $r = 2$, $l = 2$, which lead essentially to the same situation as they did in Case 2. Again we reach a contradiction since $\mathscr{P}$ does not contain a singleton. Thus assume now that

(iii) $l \geq 3$. Then

$$p\sigma(\lambda_{Mi}) < p\sigma(\lambda_{mi}) < \cdots < p\sigma(\lambda_{ma}) < p\sigma(\lambda_{Ma}),$$
$$q\sigma(\kappa_{Ma}) < q\sigma(\kappa_{ma}) < \cdots < q\sigma(\kappa_{mi}) < q\sigma(\kappa_{Mi}).$$

In this case we have

$$\begin{aligned}
\lambda_{Mi} \sim \kappa_{Ma} &\quad \text{and} \quad p\sigma(\lambda_{Mi}) = q\sigma(\lambda_{Ma}),\\
\lambda_{Ma} \sim \kappa_{Mi} &\quad \text{and} \quad p\sigma(\lambda_{Ma}) = q\sigma(\kappa_{Mi}),\\
\lambda_{mi} \sim \kappa_{ma} &\quad \text{and} \quad p\sigma(\lambda_{mi}) = q\sigma(\kappa_{ma}),\\
\lambda_{ma} \sim \kappa_{mi} &\quad \text{and} \quad p\sigma(\lambda_{ma}) = q\sigma(\kappa_{mi}).
\end{aligned}$$

These four relations in turn yield

$$\begin{aligned} plv_1 &= qkv_r\,, \\ plv_r &= qkv_1\,, \\ p((l-1)v_1 + v_2) &= q((k-1)v_r + v_{r-1})\,, \\ p((l-1)v_r + v_{r-1}) &= q((k-1)v_1 + v_2). \end{aligned}$$

The first two equations imply $v_1^2 = v_r^2$, therefore, $v_r = -v_1$ and then $pl = -qk$. Substitution into the third and fourth equations yields

$$\begin{aligned} p(v_2 - v_1) &= q(v_{r-1} - v_r)\,, \\ p(v_{r-1} - v_r) &= q(v_2 - v_1)\,; \end{aligned}$$

therefore, $p^2 = q^2$, so that $p = -q$, and finally $l = k$, which is impossible.

6. Proof of the theorems

In view of the proposition, for partitions $\mathscr{P}$ of the set T that do not contain a singleton, Theorem A and Theorem B respectively do provide everything that is needed for the proof of our assertions. As for the remaining cases, we have

Lemma 2. *Suppose that $\mathscr{P}$ contains a singleton. Then $\mathfrak{S}(\mathscr{P})$ is finite. If moreover $\{u_n\}$ satisfies the hypotheses of Theorem 2, then we have*

$$|\mathfrak{S}(\mathscr{P})| < 2klt(8sd!)^{2^{41(t+1)!d!}s^6}.$$

Proof. Suppose that $\{\lambda\} = \{(i_1, \dots, i_l)\} \in \mathscr{P}$. Then, with the notation of §3, we get

$$F_\lambda(n)\varphi_\lambda^n = 0. \tag{6.1}$$

Since $\varphi_\lambda \neq 0$ and F_λ is a nonzero polynomial, there are only finitely many such n. On the other hand given n_0, equation (1.1) becomes $bu_m^k = au_{n_0}^l$ and thus reduces to not more than k equations, each of the type

$$u_m = c\,, \tag{6.2}$$

where c is a constant; but by the Skolem-Mahler-Lech Theorem, equation (6.2) has only finitely many solutions m if $\{u_m\}$ is nondegenerate. If $\mathscr{P}$ contains a singleton $\{\kappa\} = \{(j_1, \dots, j_k)\}$, the situation is completely analogous. It follows that $\mathfrak{S}(\mathscr{P})$ is finite.

We now turn to the estimate in the algebraic case. By (1.3) and (1.4), equation (6.1) has not more than $l \cdot t$ solutions n. Given n_0, equation (1.1) is

$$u_m^k = \frac{a}{b}\, u_{n_0}^l = c_0 \in \mathscr{K}\,, \tag{6.3}$$

say. Since $u_m \in \mathscr{K}$, we may infer that there exist fixed elements $c_1, \dots, c_k \in \mathscr{K}$ (which are not necessarily distinct) such that each solution m of (6.3) satisfies at least one of the relations

$$u_m = c_j \qquad (j = 1, \dots, k). \tag{6.4}$$

Now it follows from Theorem 1.2 of Schlickewei [5] that (6.4) has not more than $2 \cdot (8sd!)^{2^{41(t+1)!d!}s^6}$ solutions $m \in \mathbb{Z}$. Allowing a factor $l \cdot t$ for the number of possible values n and a factor k for the number of possible equations (6.4), we finally get the bound

$$|\mathfrak{S}(\mathscr{P})| \leq 2klt(8sd!)^{2^{41(t+1)!d!}s^6}$$

as asserted. It is easily seen that the case of a singleton $\{\kappa\}$ implies the same estimate.

As for the proof of Theorem 1, the rank 1 case was already treated in §4. If the rank r is larger than 1, then in view of Theorem A, the proposition, and Lemma 2 for each partition $\mathscr{P}$ of T, the set $\mathfrak{S}(\mathscr{P})$ is finite, and Theorem 1 follows.

We now turn to the proof of Theorem 2. For partitions $\mathscr{P}$ that contain a singleton, Lemma 2 provides an upper bound for the cardinality $|\mathfrak{S}(\mathscr{P})|$. In view of the proposition, for all other partitions $\mathscr{P}$ of T we may apply Theorem B to equation (3.2) with parameters $N = 2$, $\delta < tk$, $D = \binom{2+\delta}{2}$, $s' = s$, $h = |T| < 2^{t+k}$ (cf. (3.1)). We may infer that for partitions $\mathscr{P}$ that do not contain a singleton,

$$|\mathfrak{S}(\mathscr{P})| < 2^{320+2\cdot s^7 2^{43d!(4^{t+k})!}}. \tag{6.5}$$

It is clear that this bound by far exceeds the bound in Lemma 2.

By Polya-Szegö [3, Abschnitt I, Kap. 1, §1, Nr. 21], the number of possible partitions $\mathscr{P}$ of T is $< 2^{2^{2(t+k)}}$. Combining this with (6.5) we get the assertion of Theorem 2.

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