ON THE AFFINE SURFACE AREA

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Abstract. It is shown that at least two expressions that extend the definition of the affine surface area to all convex bodies coincide.

1. Introduction

In the monograph [2] the affine surface area of a convex body $C$ in $\mathbb{R}^3$ with sufficiently smooth boundary is introduced by $\int_{\partial C} \kappa(x)^{1/4} d\mu(x)$ where $\kappa(x)$ is the Gauss-Kronecker curvature and $\mu$ is the surface measure on $\partial C$. It is then shown that this expression equals

$$\lim_{\delta \to 0} \frac{\sqrt{\pi} \text{vol}_3(C) - \text{vol}_3(C[\delta])}{\sqrt{\delta}};$$

$C[\delta]$ denotes the floating body of $C$: Every supporting hyperplane of $C[\delta]$ cuts off a set of volume $\delta$ from $C$. It was shown by Leichtweiß [4] that these expressions generalize in the case of higher dimensions to

$$\int_{\partial C} \kappa(x)^{1/(n+1)} d\mu(x),$$

(1)

$$\lim_{\delta \to 0} c_n \frac{\text{vol}_n(C) - \text{vol}_n(C[\delta])}{\delta^{2/(n+1)}}$$

(2)

where $c_n = 2(\text{vol}_{n-1}(B_2^{n-1}(0,1))/(n+1))^{2/(n+1)}$, provided that $C$ has a $C^2$-boundary and $\kappa(x)$ is always positive. Leichtweiß also showed that these expressions are equal. The expressions (1) and (2) do not exist for all convex bodies. Therefore, Leichtweiß suggested the following [5] as the definition for the affine surface area:

$$\lim_{\varepsilon \to 0} \lim_{\delta \to 0} n c_n \delta^{-2/(n+1)} (\text{vol}_n(C + B_2^n(0, \varepsilon)) - V((C + B_2^n(0, \varepsilon)), (C + B_2^n(0, \varepsilon))[\delta])),$$

(3)

where $V(\ldots)$ denotes the mixed volume.
At the same time Lutwak [8] gave the following as the definition for the affine surface area:

\[
\inf_{L \in S^n} \left\{ \left( \int_{\partial B^2_r} \frac{1}{\rho_L(\xi)} \, dS_C(\xi) \right) (n \vol_n(L))^{1/n} \right\}^{n/(n+1)}
\]

where \( L \) is a star body and \( \rho_L \) its radius.

Leichtweiß [6, 7] proved that (3) is smaller than or equal to (4). It is conjectured that both expressions are equal. In [11] the convex floating body \( C_\delta \) was studied, i.e., the intersection of all halfspaces \( H^+ \) with \( \vol_n(C \cap H^-) = \delta \). Clearly \( C_\delta \) exists for all \( C \) and \( \delta \) and is equal to the floating body whenever it exists. It was shown that

\[
\int_{\partial C} \kappa(x)^{1/(n+1)} \, d\mu(x) = \lim_{\delta \to 0} c_n \frac{\vol_n(C) - \vol_n(C_\delta)}{\delta^{2/(n+1)}}
\]

where \( \kappa(x) \) denotes the generalized Gauss-Kronecker curvature [10, p. 25]. A convex function \( \Phi \) on an open subset of \( \mathbb{R}^n \) is said to be twice differentiable in a generalized sense at \( x_0 \) if there is a linear map \( d^2\Phi(x_0) \) from \( \mathbb{R}^n \) into itself so that we have for all \( x \) in a neighborhood \( U(x_0) \) and all subdifferentials \( \partial\Phi(x_0) \)

\[
||d\Phi(x) - d\Phi(x_0) - d^2\Phi(x_0)(x - x_0)||^2 \leq C(||x - x_0||^2)||x - x_0||^2,
\]

where \( C \) is a function with \( \lim_{t \to 0} C(t) = 0 \). As curvature radius we take the product of the principal axes of the ellipsoid or ellipsoidal cylinder generated by \( d^2\Phi(x_0) \). It follows that (3) equals

\[
\lim_{\delta \to 0} \lim_{\varepsilon \to 0} c_n \delta^{-2/(n+1)}(\vol_n(C + B^n_2(0, \varepsilon)) - \vol_n((C + B^n_2(0, \varepsilon))_{[\delta]})).
\]

We show that the expressions (3) and (5) are equal. Then we show that (5) and, thus, (3) are valuations, a question raised by Leichtweiß [6].

2. Preliminaries

The \( n \)-dimensional volume \( \vol_n(A) \) of a subset \( A \) of \( \mathbb{R}^n \) is the Lebesgue measure, and the \( (n-1) \)-dimensional volume \( \vol_{n-1}(A) \) is the \( (n-1) \)-dimensional Hausdorff measure of \( A \). The surface measure on the boundary of a convex set is the restriction of the \( (n-1) \)-dimensional Hausdorff measure to the boundary. We also note that the Hausdorff measure is Borel regular [3]. \( B^n_2(x, r) \) denotes the Euclidean ball with radius \( r \) and center \( x \) in \( \mathbb{R}^n \).

A convex surface is almost everywhere twice differentiable in a generalized sense [1]. As a consequence the indicatrix of Dupin exists almost everywhere, and thus we can define a generalized Gauss-Kronecker curvature \( \kappa(x) \) that exists almost everywhere [10].

For every \( x \) in the boundary \( \partial C \) of a convex body \( C \) that has a unique normal we define \( \Delta(C, x, \delta) \) or \( \Delta(x, \delta) \) to be the width of a slice of volume \( \delta \) whose defining hyperplane is orthogonal to the normal at \( x \). We have [11]

\[
\kappa(x) = \lim_{\delta \to 0} c_n \frac{\Delta(x, \delta)}{\delta^{2/(n+1)}}
\]

where \( c_n \) is as in (2).
For a convex body $C$ in $\mathbb{R}^n$ the nearest point projection $q$ from $\mathbb{R}^n$ onto $C$ is defined by $\|q(x) - x\|_2 = \inf_{y \in C} \|y - x\|_2$. Let $\widetilde{C}$ be a convex body containing $C$, and let $p$ be the restriction of $q$ to $\partial \widetilde{C}$. Then we have for all Borel subsets $A$ of $\partial \widetilde{C}$ that

\[\text{vol}_{n-1}(p(A)) \leq \text{vol}_{n-1}(A).\]

A cap of $C$ at $x$ with height $h$ is denoted by $\text{cap}(C, x, h)$.

### 3. The equality of (3) and (5)

**Proposition 1.** The expressions (3) and (5) are equal.

Proposition 1 follows from the next lemma. One has to use that $C_\delta$ and $C_{[\delta]}$ coincide whenever $C_{[\delta]}$ exists.

**Lemma 2.** Let $C$ be a convex body in $\mathbb{R}^n$. Then we have

\[\lim_{\delta \to 0} \frac{\text{vol}_n(C) - \text{vol}_n(C_\delta)}{\delta^{2/(n+1)}} = \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \delta^{-2/(n+1)} \{\text{vol}_n(C + B^n_2(0, \varepsilon)) - \text{vol}_n((C + B^n_2(0, \varepsilon))_\delta)\}.\]

**Proof of Lemma 2.** We show first that the right-hand expression of (8) is smaller than the left-hand expression. We have a.e.

\[\Delta(C + B^n_2(0, \varepsilon), x, \delta) \leq \Delta(C, p(x), \delta)\]

where $p$ is the restriction of the nearest point projection from $\partial(C + B^n_2(0, \varepsilon))$ to $\partial C$. Equation (9) follows from

\[\text{cap}(C, p(x), h) + \varepsilon N(x) \subseteq \text{cap}(C + B^n_2(0, \varepsilon), x, h).\]

If a convex body $C$ in $\mathbb{R}^n$ contains a Euclidean ball of radius $r$ then

\[\text{vol}_{n-1}(\partial C) \leq \text{vol}_{n-1}(\partial(C + B^n_2(0, \varepsilon))) \leq (1 + \varepsilon/r)^{n-1} \text{vol}_{n-1}(\partial C)\]

because $C \subseteq C + B^n_2(0, \varepsilon) \subseteq (1 + \varepsilon/r)C$.

Let $A_j$ be measurable subsets of $\partial(C + B^n_2(0, \varepsilon))$ and $a_j \geq 0$ so that

\[\sum_{j=1}^N a_j \chi_{A_j}(x) \leq \lim_{\delta \to 0} c_n \delta^{-2/(n+1)} \Delta(C + B^n_2(0, \varepsilon), x, \delta)\]

holds almost everywhere and

\[\int_{\partial(C + B^n_2(0, \varepsilon))} \lim_{\delta \to 0} c_n \delta^{-2/(n+1)} \Delta(C + B^n_2(0, \varepsilon), x, \delta) \, d\mu_{C + B^n_2(0, \varepsilon)} \]

\[\leq \int_{\partial(C + B^n_2(0, \varepsilon))} \sum_{j=1}^N a_j \chi_{A_j} \, d\mu_{C + B^n_2(0, \varepsilon)}.\]
Then we get
\[
(1 - \eta) \int_{\partial(C + B_2^n(0, \varepsilon))} \lim_{\delta \to 0} c_n \delta^{-2/(n+1)} \Delta(C + B_2^n(0, \varepsilon), x, \delta) \, d\mu_{C + B_2^n(0, \varepsilon)}
\]
\[
\leq \sum_{j=1}^N a_j \mu_{C + B_2^n(0, \varepsilon)}(A_j)
\]
\[
= \sum_{j=1}^N a_j \mu_C(p(A_j)) + \sum_{j=1}^N a_j(\mu_{C + B_2^n(0, \varepsilon)}(A_j) - \mu_C(p(A_j))).
\]
By (9) and
\[
\Delta(B_2^n(0, \varepsilon), (\varepsilon, 0, \ldots, 0), \delta) \leq \Delta(C + B_2^n(0, \varepsilon), x, \delta)
\]
we get that the last expression is smaller than
\[
\int_{\partial C} \lim_{\delta \to 0} c_n \delta^{-2/(n+1)} \Delta(C, y, \delta) \, d\mu_C(y)
\]
\[
+ \varepsilon^{-(n-1)/(n+1)} \left( \text{vol}_{n-1}(\partial(C + B_2^n(0, \varepsilon))) - \text{vol}_{n-1}(\partial C) \right).
\]
Because of (10), the second summand can be estimated by
\[
\varepsilon^{-(n-1)/(n+1)}((1 + \varepsilon/r)^{n-1} - 1) \text{vol}_{n-1}(\partial C).
\]
Therefore, we get altogether
\[
\limsup_{\varepsilon \to 0} \int_{\partial(C + B_2^n(0, \varepsilon))} \lim_{\delta \to 0} c_n \delta^{-2/(n+1)} \Delta(C + B_2^n(0, \varepsilon), x, \delta) \, d\mu_{C + B_2^n(0, \varepsilon)}
\]
\[
\leq \int_{\partial C} \lim_{\delta \to 0} c_n \delta^{-2/(n+1)} \Delta(C, y, \delta) \, d\mu_C.
\]
In view of (6) we may plug in \( \kappa(x) \).

In order to show that the right-hand side of (8) is larger than the left-hand side we require a lemma.

**Lemma 3.** Let \( x \in \partial(C + B_2^n(0, \varepsilon)) \), and suppose that the indicatrix of Dupin at \( p(x) \in \partial C \) is an ellipsoid with radius \( R = (R_1, \ldots, R_{n-1}) \). Then we have
\[
\kappa(\partial(C + B_2^n(0, \varepsilon)), x) = \prod_{i=1}^{n-1} (R_i(p(x)) + \varepsilon)^{-1}.
\]

The set \( \{ y \in \partial C | \kappa(y) > 0 \} \) is measurable since \( \kappa(y)^{1/(n+1)} \in L^1(\partial C) \) [11]. Since the Hausdorff measure is Borel regular, there is a subset \( A \) of \( \{ y \in \partial C | \kappa(y) > 0 \} \) that is a Borel set having the same measure. By Lemma 3 we obtain
\[
\int_{\partial(C + B_2^n(0, \varepsilon))} \kappa(x)^{1/(n+1)} \, d\mu_{C + B_2^n(0, \varepsilon)}
\]
\[
\geq \int_{p^{-1}(A)} \prod_{i=1}^{n-1} (R_i(p(x)) + \varepsilon)^{-1/(n+1)} \, d\mu_{C + B_2^n(0, \varepsilon)}.
\]
As above we get that the last expression is larger than or equal to

\[ \int_A \prod_{i=1}^{n-1} (R_i(y) + \varepsilon)^{-1/(n+1)} d\mu_C. \]

Applying Fatou's lemma we get

\[
\liminf_{\varepsilon \to 0} \int_{\partial(C+B_2^n(0,\varepsilon))} \kappa(x)^{1/(n+1)} d\mu_{C+B_2^n(0,\varepsilon)} \geq \int_{\partial C} \kappa(y)^{1/(n+1)} d\mu_C. \quad \square
\]

4. THE AFFINE SURFACE AREA IS A VALUATION

A map \( T \) from the family of convex bodies into \( \mathbb{R} \) is called a valuation if

\[ T(K \cup L) + T(K \cap L) = T(K) + T(L) \]

whenever \( K \cup L \) is convex.

**Proposition 4.** The affine surface area is a valuation.

**Lemma 5.** Let \( K \) and \( L \) be convex bodies in \( \mathbb{R}^n \), and suppose that \( K \cup L \) is a convex body. Then we have for all \( x \in \partial K \cap \partial L \) where all the curvatures \( \kappa_{K \cup L}, \kappa_{K \cap L}, \kappa_K, \) and \( \kappa_L \) exist that

\[
\kappa_{K \cup L}(x) = \min\{\kappa_K(x), \kappa_L(x)\},
\]

\[
\kappa_{K \cap L}(x) = \max\{\kappa_K(x), \kappa_L(x)\}.
\]

Please note that the set where one of the curvatures does not exist is a null set [10].

For the proof of Lemma 5 we only have to observe that the indicatrix of Dupin of \( K \cup L \) at \( x \) is the union of those of \( K \) and \( L \) at \( x \). Moreover, the indicatrix of \( K \cap L \) at \( x \) is the intersection of those of \( K \) and \( L \). Then one uses that the intersection or union of two ellipsoids is again an ellipsoid if and only if one ellipsoid is contained in the other.

**Proof of Proposition 4.** The affine surface area of a convex body \( M \) equals \( \int_{\partial M} \kappa_M(x)^{1/(n+1)} d\mu_M \). We apply this formula to the bodies \( K \cup L, K \cap L, K, \) and \( L, \) and decompose the surfaces

\[
\partial(K \cup L) = (\partial K \cap \partial L) \cup (\partial K \cap L^c) \cap (\partial L \cap K^c),
\]

\[
\partial(K \cap L) = (\partial K \cap \partial L) \cup (\partial K \cap L^c) \cup (\partial L \cap K^c),
\]

\[
\partial K = (\partial K \cap \partial L) \cup (\partial K \cap L^c) \cup (\partial L \cap K^c),
\]

\[
\partial L = (\partial K \cap \partial L) \cup (\partial L \cap K^c) \cup (\partial L \cap K^c),
\]

where \( K^c \) is the complement of \( K \) and \( \overset{\bullet}{K} \) is the interior of \( K \).

Since all sets (except possibly \( \partial K \cap \partial L \)) are open subsets of \( \partial K, \partial L, \partial(K \cap L), \) and \( \partial(K \cup L) \) and since the curvature is a local invariant, the
integrals over those sets cancel out. It remains to show
\[
\int_{\partial K \cap \partial L} \kappa_{K \cup L}(x) \, d\mu_{K \cup L} + \int_{\partial K \cap \partial L} \kappa_{K \cap L}(x) \, d\mu_{K \cap L} = \int_{\partial K \cap \partial L} \kappa_K(x) \, d\mu + \int_{\partial K \cap \partial L} \kappa_L(x) \, d\mu.
\]
This follows from Lemma 5. □

REFERENCES


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