A CONSTRUCTION OF A SUBSPACE IN EUCLIDEAN SPACE
WITH DESIGNATED VALUES OF DIMENSION
AND METRIC DIMENSION

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Dedicated to Professor Ryosuke Nakagawa on his 60th birthday

Abstract. For every integer $m$, $k$, and $n$ such that $0 < m < n - 1$ and $m < k < \min\{2m, n - 1\}$, we construct a subspace $S^n_{m,k}$ in Euclidean $n$-space $\mathbb{R}^n$ satisfying the conditions that $\mu\dim S^n_{m,k} = m$ and $\dim S^n_{m,k} = k$, where $\mu\dim$ denotes the metric dimension.

1. Introduction

Let $X$ be a subspace in Euclidean $n$-space $\mathbb{R}^n$ with $\mu\dim X = m$, $0 < m \leq n - 1$, where $\mu\dim$ denotes the metric dimension. Then we have $m \leq \dim X \leq \min\{2m, n - 1\}$ by Katetov's inequality $\dim X \leq 2\mu\dim X$ [3].

In a previous paper [2] we constructed a subspace $S^n_{n,m}$ in $\mathbb{R}^n$ such that $\mu\dim S^n_{n,m} = m$ and $\dim S^n_{n,m} = \min\{2m, n - 1\}$. Thus the space $S^n_{n,m}$ is a subspace in $\mathbb{R}^n$ of metric dimension $m$ which has the maximal discrepancy with its covering dimension.

The purpose of this note is to prove the following theorem.

Theorem. Let $m$, $k$, and $n$ be arbitrary integers such that $0 < m < n - 1$ and $m < k < \min\{2m, n - 1\}$. Then there exists a subspace $S^n_{m,k}$ in $\mathbb{R}^n$ such that $\mu\dim S^n_{m,k} = m$ and $\dim S^n_{m,k} = k$.

The space $S^n_{m,k}$ given below can be expressed as $S^n_{m,k} = S^n_m \cap N^n_k$, where $S^n_m$ denotes a space which is a slight modification of $S^n_{n,m}$ and $N^n_k$ denotes Nöbeling's $k$-dimensional space in $\mathbb{R}^n$.

2. Notation and definitions

We denote by $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$ the set of rational numbers, integers, and positive integers, respectively. For a point $x = (x_i)$ in $\mathbb{R}^n$ we let $r(x) = \text{card}\{i: x_i \in \mathbb{Q}\}$. Then Nöbeling's $k$-dimensional space $N^n_k$ in $\mathbb{R}^n$ can be expressed as $N^n_k = \{x \in \mathbb{R}^n : r(x) \leq k\}$ (cf. [1]).

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For a metric space $X$ the metric dimension $\mu \dim X$ is defined as follows: $\mu \dim X \leq m$ iff for every $\epsilon > 0$ there exists an open cover $\mathcal{U}$ of $X$ such that $\text{mesh} \mathcal{U} < \epsilon$ and $\text{ord} \mathcal{U} \leq n + 1$ (cf. [1]).

The construction of the space $S_{n, m}$ in [2] is as follows. Let $T_i = \{t_{i,j} : j \in \mathbb{Z}\}, i \in \mathbb{N}$, be a set of real numbers such that:

(1) 
$$t_{i,j+1} - t_{i,j} = 1/i$$

for every $j$ and

(2) 
$$T_i \cap T_{i'} = \emptyset \text{ if } i \neq i'.$$

For every $x \in \mathbb{R}^n$, we set $r_i(x) = \text{card}\{j : x_j \in T_i\}, i \in \mathbb{N}$. Then the space $S_{n, m}$ in [2] can be written as

$$S_{n, m} = \{x \in \mathbb{R}^n : r_i(x) \leq m \text{ for every } i \in \mathbb{N}\}.$$

In this construction we can assume that each $T_i$ is contained in $\mathbb{Q}$, and we denote by $S^n_m$ the space $S_{n, m}$ obtained by this modification.

3. Proof of the Theorem

We need a lemma due to Wilkinson [4].

Lemma. Suppose that $A_1, A_2, \ldots$ are closed proper subsets in $\mathbb{R}^n$ such that $\dim(A_i \cap A_j) \leq m$ whenever $i \neq j$. Then we have $\dim(\mathbb{R}^n - \bigcup_{i=1}^{\infty} A_i) \geq n - m - 2$.

Proof of the Theorem. By the definition

$$S^n_m = \{x \in \mathbb{R}^n : r_i(x) \leq m \text{ for every } i \in \mathbb{N}\}.$$ 

Since $T_i \subset \mathbb{Q}$ for every $i$, it follows that $r_i(x) \leq r(x)$ for $x \in \mathbb{R}^n$, and hence $N^n_m \subset S^n_m$. Thus we have $N^n_m = S^n_{m,m} \subset S^n_{m,k} \subset S^n_m$, which implies that $m = \mu \dim N^n_m \leq \mu \dim S^n_{m,k} \leq \mu \dim S^n_m = m$ by [2, Lemma 4]. Thus we obtain

$$\mu \dim S^n_{m,k} = m \text{ for every } k, \ m \leq k \leq \min\{2m, n - 1\}.$$ 

Moreover, we have $\dim(S^n_{m,k+1} - S^n_{m,k}) \leq 0$ because $S^n_{m,k+1} - S^n_{m,k} = S^n_m \cap (N^n_{k+1} - N^n_k)$ and $\dim(N^n_{k+1} - N^n_k) = 0$. Thus we obtain

$$\dim S^n_{m,k+1} \leq \dim S^n_{m,k} + 1 \text{ for every } k, \ m \leq k \leq \min\{2m, n - 1\}.$$ 

We let $A_i = \{x \in \mathbb{R}^n : r_i(x) \geq m + 1\}, i \in \mathbb{N}$. Condition (1) implies that each $A_i$ is the union of a countable locally finite family of $(n - m - 1)$-dimensional planes. Hence, $A_i$ is closed in $\mathbb{R}^n$ and $\dim A_i = n - m - 1$. From (3) it follows that $\mathbb{R}^n - S^n_m = \bigcup\{A_i : i \in \mathbb{N}\}$.

Case 1. $2m \geq n - 1$. In this case we have a sequence

$$N^n_m = S^n_{m,m} \subset S^n_{m,m+1} \subset \cdots \subset S^n_{m,n-1}.$$ 

In view of (5), to prove that $\dim S^n_{m,k} = k$ for every $k$ it suffices to show

$$\dim S^n_{m,n-1} \geq n - 1.$$ 

From (2) and the assumption that $2m \geq n - 1$, it follows that $A_i \cap A_j = \emptyset$ if $i \neq j$. Since $\mathbb{R}^n - N^n_{n-1}$ consists of a countable number of points and since

$$\mathbb{R}^n - S^n_{m,n-1} = (\mathbb{R}^n - S^n_m) \cup (\mathbb{R}^n - N^n_{n-1}) = \bigcup\{A_i : i \in \mathbb{N}\} \cup (\mathbb{R}^n - N^n_{n-1}),$$

we obtain (6) by the Lemma.
Case 2. $2m < n - 1$. In this case we have a sequence

$$N^n_m = S^n_{m,m} \subset S^n_{m,m+1} \subset \cdots \subset S^n_{m,2m}.$$ 

It follows from (2) that $\dim(A_i \cap A_j) \leq n - 2m - 2$ if $i \neq j$, because $2m < n - 1$. On the other hand, $\mathbb{R}^n - N^n_{2m}$ is the countable union of $(n - 2m - 1)$-dimensional planes $B_j$, $j \in \mathbb{N}$. Hence we have

$$\mathbb{R}^n - S^n_{m,k} = \bigcup \{A_i: i \in \mathbb{N}\} \cup \{B_j: j \in \mathbb{N}\}.$$ 

It is clear that, for every $i$ and $j$, $B_j \subset A_i$, or $\dim(A_i \cap B_j) \leq n - 2m - 2$ and that $\dim(B_j \cap B_{j'}) \leq n - 2m - 2$ whenever $j \neq j'$. Thus it follows from the Lemma that

$$\dim S^n_{m,2m} \geq n - (n - 2m - 2) - 2 = 2m.$$ 

This implies that $S^n_{m,k} = k$ for every $k$, $m \leq k \leq 2m$, by virtue of (5). This completes the proof.

References


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