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A CONSTRUCTION OF A SUBSPACE IN EUCLIDEAN SPACE WITH DESIGNATED VALUES OF DIMENSION AND METRIC DIMENSION

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Dedicated to Professor Ryosuke Nakagawa on his 60th birthday

ABSTRACT. For every integer m, k, and n such that $0 \leq m \leq n-1$ and $m \leq k \leq \min\{2m, n-1\}$, we construct a subspace $S^n_{m,k}$ in Euclidean n-space $\mathbf{R}^n$ satisfying the conditions that $\mu\dim S^n_{m,k} = m$ and $\dim S^n_{m,k} = k$, where $\mu\dim$ denotes the metric dimension.

1. INTRODUCTION

Let X be a subspace in Euclidean n-space $\mathbf{R}^n$ with $\mu\dim X = m$, $0 \leq m \leq n-1$, where $\mu\dim$ denotes the metric dimension. Then we have $m \leq \dim X \leq \min\{2m, n-1\}$ by Katětov's inequality $\dim X \leq 2\mu\dim X$ [3].

In a previous paper [2] we constructed a subspace $S_{n,m}$ in $\mathbf{R}^n$ such that $\mu\dim S_{n,m} = m$ and $\dim S_{n,m} = \min\{2m, n-1\}$. Thus the space $S_{n,m}$ is a subspace in $\mathbf{R}^n$ of metric dimension m which has the maximal discrepancy with its covering dimension.

The purpose of this note is to prove the following theorem.

Theorem. *Let m, k, and n be arbitrary integers such that $0 \leq m \leq n-1$ and $m \leq k \leq \min\{2m, n-1\}$. Then there exists a subspace $S^n_{m,k}$ in $\mathbf{R}^n$ such that $\mu\dim S^n_{m,k} = m$ and $\dim S^n_{m,k} = k$.*

The space $S^n_{m,k}$ given below can be expressed as $S^n_{m,k} = S^n_m \cap N^n_k$, where S^n_m denotes a space which is a slight modification of $S_{n,m}$ and N^n_k denotes Nöbeling's k-dimensional space in $\mathbf{R}^n$.

2. NOTATION AND DEFINITIONS

We denote by $\mathbf{Q}, \mathbf{Z}$, and $\mathbf{N}$ the set of rational numbers, integers, and positive integers, respectively. For a point $x = (x_i)$ in $\mathbf{R}^n$ we let $r(x) = \operatorname{card}\{i : x_i \in \mathbf{Q}\}$. Then Nöbeling's k-dimensional space N^n_k in $\mathbf{R}^n$ can be expressed as $N^n_k = \{x \in \mathbf{R}^n : r(x) \leq k\}$ (cf. [1]).

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For a metric space X the metric dimension $\mu \dim X$ is defined as follows; $\mu\dim X \leq m$ iff for every $\varepsilon > 0$ there exists an open cover $\mathscr{U}$ of X such that $\operatorname{mesh}\mathscr{U} < \varepsilon$ and $\operatorname{ord}\mathscr{U} \leq n+1$ (cf. [1]).

The construction of the space $S_{n,m}$ in [2] is as follows. Let $T_i = \{t_{i,j}\colon j \in \mathbf{Z}\}$, $i \in \mathbf{N}$, be a set of real numbers such that:

$$t_{i,j+1} - t_{i,j} = 1/i \quad \text{for every } j \tag{1}$$

and

$$T_i \cap T_{i'} = \varnothing \quad \text{if } i \neq i'. \tag{2}$$

For every $x \in \mathbf{R}^n$, we set $r_i(x) = \operatorname{card}\{j\colon x_j \in T_i\}$, $i \in \mathbf{N}$. Then the space $S_{n,m}$ in [2] can be written as

$$S_{n,m} = \{x \in \mathbf{R}^n\colon r_i(x) \leq m \text{ for every } i \in \mathbf{N}\}.$$

In this construction we can assume that each T_i is contained in $\mathbf{Q}$, and we denote by S^n_m the space $S_{n,m}$ obtained by this modification.

3. Proof of the Theorem

We need a lemma due to Wilkinson [4].

Lemma. *Suppose that $A_1, A_2, \ldots$ are closed proper subsets in $\mathbf{R}^n$ such that $\dim(A_i \cap A_j) \leq m$ whenever $i \neq j$. Then we have* $\dim(\mathbf{R}^n - \bigcup_{i=1}^{\infty} A_i) \geq n-m-2$.

Proof of the Theorem. By the definition

$$S^n_m = \{x \in \mathbf{R}^n\colon r_i(x) \leq m \text{ for every } i \in \mathbf{N}\}. \tag{3}$$

Since $T_i \subset \mathbf{Q}$ for every i, it follows that $r_i(x) \leq r(x)$ for $x \in \mathbf{R}^n$, and hence $N^n_m \subset S^n_m$. Thus we have $N^n_m = S^n_{m,m} \subset S^n_{m,k} \subset S^n_m$, which implies that $m = \mu\dim N^n_m \leq \mu\dim S^n_{m,k} \leq \mu\dim S^n_m = m$ by [2, Lemma 4]. Thus we obtain

$$\mu\dim S^n_{m,k} = m \quad \text{for every } k, \ m \leq k \leq \min\{2m, n-1\}. \tag{4}$$

Moreover, we have $\dim(S^n_{m,k+1} - S^n_{m,k}) \leq 0$ because $S^n_{m,k+1} - S^n_{m,k} = S^n_m \cap (N^n_{k+1} - N^n_k)$ and $\dim(N^n_{k+1} - N^n_k) = 0$. Thus we obtain

$$\dim S^n_{m,k+1} \leq \dim S^n_{m,k} + 1 \quad \text{for every } k, \ m \leq k \leq \min\{2m, n-1\}. \tag{5}$$

We let $A_i = \{x \in \mathbf{R}^n\colon r_i(x) \geq m+1\}$, $i \in \mathbf{N}$. Condition (1) implies that each A_i is the union of a countable locally finite family of $(n-m-1)$-dimensional planes. Hence, A_i is closed in $\mathbf{R}^n$ and $\dim A_i = n-m-1$. From (3) it follows that $\mathbf{R}^n - S^n_m = \bigcup\{A_i\colon i \in \mathbf{N}\}$.

Case 1. $2m \geq n-1$. In this case we have a sequence

$$N^n_m = S^n_{m,m} \subset S^n_{m,m+1} \subset \cdots \subset S^n_{m,n-1}.$$

In view of (5), to prove that $\dim S^n_{m,k} = k$ for every k it suffices to show

$$\dim S^n_{m,n-1} \geq n-1. \tag{6}$$

From (2) and the assumption that $2m \geq n-1$, it follows that $A_i \cap A_j = \varnothing$ if $i \neq j$. Since $\mathbf{R}^n - N^n_{n-1}$ consists of a countable number of points and since

$$\mathbf{R}^n - S^n_{m,n-1} = (\mathbf{R}^n - S^n_m) \cup (\mathbf{R}^n - N^n_{n-1}) = \bigcup\{A_i\colon i \in \mathbf{N}\} \cup (\mathbf{R}^n - N^n_{n-1}),$$

we obtain (6) by the Lemma.

Case 2. $2m < n-1$. In this case we have a sequence

$$N_m^n = S_{m,m}^n \subset S_{m,m+1}^n \subset \cdots \subset S_{m,2m}^n.$$

It follows from (2) that $\dim(A_i \cap A_j) \leq n-2m-2$ if $i \neq j$, because $2m < n-1$. On the other hand, $\mathbf{R}^n - N_{2m}^n$ is the countable union of $(n-2m-1)$-dimensional planes B_j, $j \in \mathbf{N}$. Hence we have

$$\mathbf{R}^n - S_{m,k}^n = \bigcup\{A_i : i \in \mathbf{N}\} \cup \{B_j : j \in \mathbf{N}\}.$$

It is clear that, for every i and j, $B_j \subset A_i$ or $\dim(A_i \cap B_j) \leq n-2m-2$ and that $\dim(B_j \cap B_{j'}) \leq n-2m-2$ whenever $j \neq j'$. Thus it follows from the Lemma that

$$\dim S_{m,2m}^n \geq n-(n-2m-2)-2 = 2m.$$

This implies that $S_{m,k}^n = k$ for every k, $m \leq k \leq 2m$, by virtue of (5). This completes the proof.

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