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THE HAUSDORFF DIMENSION OF THE NONDIFFERENTIABILITY SET OF THE CANTOR FUNCTION IS $[\ln(2)/\ln(3)]^2$

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ABSTRACT. The main purpose of this note is to verify that the Hausdorff dimension of the set of points N^* at which the Cantor function is not differentiable is $[\ln(2)/\ln(3)]^2$. It is also shown that the image of N^* under the Cantor function has Hausdorff dimension $\ln(2)/\ln(3)$. Similar results follow for a standard class of Cantor sets of positive measure and their corresponding Cantor functions.

The Hausdorff dimension of the set of points N^* at which the Cantor function is not differentiable is $[\ln(2)/\ln(3)]^2$.

Chapter 1 in [5] provides a nice introduction to Hausdorff measure and dimension; references [5–7] pursue the topic. We begin our proof with some notation and discussion. Let C denote the Cantor set. Let N^+ (N^-) denote the set of points at which the Cantor function does not have a right side (left side) derivative, finite or infinite. Then $N^* = N^+ \cup N^- \cup \{t : t$ is an end point of $C\}$ denotes the nondifferentiability set of the Cantor function. Although we will assume familiarity with [4], where Eidswick characterized N^*, some material is repeated for completeness.

A number t in C has a ternary representation $t = (t_1, \dots, t_i, \dots)$, where $t_i = 0$ or 2.

Let $z(n)$ denote the position of the nth zero in the ternary representation of t;

(1a) If $t \in N^+$, then $\limsup\{z(n+1)/z(n)\} \geq \ln(3)/\ln(2)$;
(1b) If $\limsup\{z(n+1)/z(n)\} > \ln(3)/\ln(2)$, then $t \in N^+$.

Let m_d denote the d-dimensional Hausdorff measure, and put $r = \ln(2)/\ln(3)$.

We will compute the Hausdorff dimension of N^* by verifying

(A) If $1 \geq d > r^2$, then $m_d N^* = 0$.
(B) If $d < r^2$, then $m_d N^* \geq K_d > 0$; K_d will be specified later for a sequence of d's increasing to r^2.

Condition (A) will be verified for each d satisfying the inequalities $1 \geq d > r^2$ by constructing a set E (depending on d) which contains N^* and satisfies the equation $m_d E = 0$. To verify (B), we will consider a sequence $\{d_n\}$ of

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d's increasing to r^2; for each d in the sequence, we will construct a subset E^* of N^* with $m_d(E^*) > 0$, which implies $m_h(N^*) \geq m_h(E^*) = \infty$ for $h < d$. (A) implies $m_h(N^*) = 0$ for $h > r^2$, and (B) implies $m_h(N^*) = \infty$ for $h < r^2$. Consequently, the Hausdorff dimension of N^* is equal to r^2.

Verification of (A). We will use sets $E_k = \{t\colon t_k = 0 \text{ and } t_i = 2 \text{ for } k < i \leq u_k\}$, where u_k will be specified below.

Fix $d > r^2$. We will define a positive integer n (depending on d) and u_k for $k \geq n$ so

$$N^+ \subseteq \bigcup_{k \geq m} E_k\,,\ m \geq n\colon N^+ \subseteq \limsup\{E_k\} = E^\infty \tag{2}$$

and

$$2^k/(3^{u_k})^d \leq k^{-2}\colon k\ln(2) - du_k\ln(3) \leq -2\ln(k)\colon \\ r + (2/\ln(3))(\ln(k)/k) \leq d(u_k/k). \tag{3}$$

The required strings of 2's in the points of E^∞ will be short enough to apply (1a) to verify (2), and they will be long enough to satisfy (3). Since $d > r^2$, put $d = r(r+t)$, where $t > 0$. Then $t = (d - r^2)/r < 1/r$. Choose $n \geq 3$ so that

$$\ln(n)/n < t/4. \tag{4}$$

Then $\ln(m)/m$ is decreasing for $m \geq n$ and $1/n < t/4$. Thus, for $k \geq n$ we can choose u_k so that

$$r^{-1} - t/2 < u_k/k < r^{-1} < t/4. \tag{5}$$

According to (4) and the first inequality in (5), for $k \geq n$,

$$\begin{aligned} r + (2/\ln(3))(\ln(k)/k) &< r + 2(\ln(k)/k) < r + t/2 \\ &= r + t - t/2 \leq r + t - d(t/2) = d(r^{-1} - t/2) < d(u_k/k), \end{aligned}$$

so (3) is satisfied for $k \geq n$.

Referring to (1a) and the second inequality in (5), (2) is satisfied. To show that $m_d(\limsup\{E_k\}) = m_d(E^\infty) = 0$, it suffices to observe that since each E_i can be covered with 2^{i-1} intervals of length 3^{-u_i}, then

$$m_d(E^\infty) \leq \lim_k \sum_{i \geq k} 2^i/(3^{u_i})^d \leq \lim_k \sum_{i \geq k} i^{-2} = 0.$$

Consequently, $m_d N^+ = 0$. Similarly, $m_d N^- = 0$; thus, $m_d N^* = 0$.

Verification of (B). For $d = rsv$, where $s = (n-1)/n$ and $v = rs$, we will construct a subset $E = E_n$ of N^+ with $m_d E \geq K = PQR$, where P and Q are positive numbers that will be defined later and R is a positive constant relating m_d and the equivalent d-dimensional net measure $(\text{ter})_d$ obtained by requiring covers to be composed of ternary intervals $[a, b] = [i/3^k, (i+1)/3^k]$ (which we call k-intervals) according to the inequality $m_d \geq R(\text{ter})_d$. The existence of R follows from a variation on a theme of Besicovitch that is discussed in [5, §5.1; 7, Chapter 2, §7.1]; we are using closed ternary intervals, but only countably many end points are involved in intersections. Since $m_d \geq R(\text{ter})_d$, we verify (B) by establishing the inequality $(\text{ter})_d E \geq PQ$ below.

Covers are required to be ternary covers in the following discussion.

We begin by describing a generic set E of the type to be used; E corresponds to a sequence $0 < k_1 < u_1 < k_2 < u_2 < \cdots$ of positive integers as follows:

$$E = \{t = (t(1), t(2), \dots): t(k_i) = 0 \text{ and } t(k) = 2 \text{ for } k_i < k \leq u_i, i \geq 1\}.$$

The set E is a closed subset of C and is composed of non-end-points of C.

When $k_i \leq k \leq u_i$, k is a *fixed choice* (for E); otherwise, k is a *free choice.* The strings of fixed choices will be long enough to make the points in E satisfy (1b), and the strings of free choices will be long enough to assure $P > 0$ and $Q > 0$.

Let $F(p, q)$ denote the number of free choices k with $p < k \leq q$.

Because of [4, Theorem 1] and the fact that

$$\liminf(3^{z(n)}/2^{z(n+1)}) \leq \liminf(3^{k_i}/2^{u_i}),$$

E is a subset of N^+ if $\inf_i(u_i/k_i) > \ln 3/\ln 2$. In particular, recalling the definition of v, E is a subset of N^+ if $u_i = v^{-1}k_i + r_i$, where $0 \leq r_i < 1$.

Let $\{[a_j, b_j]\}$ be a ternary cover of E. Since E contains no end point of C, $\{(a_j, b_j)\}$ is an open cover of E; E is compact, so we restrict attention to a finite subcover. We can also require $b_j \leq a_{j+1}$ and that $[a_j, b_j] \cap E$ be nonempty. Let $3^{-w} = \min\{b_j - a_j\}$. For $k > w$, a k-interval $U = [i/3^k, (i+1)/3^k]$ intersects at most one (a_j, b_j); if this intersection is nonempty, then $U \subseteq [a_j, b_j]$.

The k_i's and u_i's considered below are all $> w$. To prove $(\text{ter})_d E \geq PQ$, it suffices to specify positive constants P and Q satisfying

(C) $m_d[a_j, b_j] \geq P$ (number of u_i-intervals in $[a_j, b_j]$)$3^{-u_i d}$

(D) (number of u_i-intervals which intersect E)$3^{-u_i d} \geq Q$.

Letting $[i/3^k, (i+1)/3^k]$ denote a generic $[a_j, b_j]$, we rewrite (C) and (D) as

(C) $3^{-kd} \geq P 2^{F(k, k_i)}/3^{u_i d}$.

(D) $2^{F(0, k_i)}/3^{u_i d} \geq Q$.

Define u_i by the equation $u_i = v^{-1}k_i + r_i$, $0 \leq r_i < 1$. Thus, the points in E satisfy (1b).

Define k_{i+1} by specifying $F(0, k_{i+1}) = sk_{i+1} + s_{i+1}$, where $0 \leq s_{i+1} < 1$ and s_{i+1} is minimal. Such a choice is possible because for $1 < f < k$,

$$f/k - (f-1)/(k-1) = (k-f)/[k(k-1)] < k^{-1}.$$

This definition of k_{i+1} provides enough free choices to assure $P > 0$ and $Q > 0$.

Verification of (C).

$$3^{-kd} \geq P 2^{F(k, k_i)}/3^{u_i d} \Leftrightarrow 3^{(u_i - k)d} \geq P 2^{F(k, k_i)}$$
$$\Leftrightarrow 2^{(u_i - k)sv} P 2^{F(k, k_i)} \Leftrightarrow 1 \geq P 2^{[F(k, k_i) - vs(u_i - k)]}.$$

Put $h(k, i) = F(k, k_i) - sv(u_i - k)$. If $k_j \leq k \leq u_j$, then $h(k, i) \leq h(u_j, i)$; and if $u_{j-1} < k < k_j$, then $h(k, i) < h(u_{j-1}, i)$. Thus, it suffices to show that $h(u_j, i)$ is bounded for $j < i$.

$$\begin{aligned} F(u_j, k_i) &- sv(u_i - u_j) \\ &= [(sk_i + s_i) - (sk_j + s_j)] - sv[(v^{-1}k_i + r_i) - (v^{-1}k_j + r_j)] \\ &= [s_i - s_j] - sv[r_i - r_j] < 1 + sv. \end{aligned}$$

Hence, we put $P = 2^{-(1+sv)}$.

Verification of (D). $F(0, k_i) = sk_i + s_i$, $u_i d = (v^{-1}k_i + r_i)rsv = rs(k_i + vr_i)$, and $3^r = 2$; consequently, $2^{F(0,k_i)}/3^{u_i d} = 2^{(s_i - svr_i)} \geq 2^{-sv}$. Thus, putting $Q = 2^{-sv}$, we have shown that the Hausdorff dimension of N^* is r^2.

Now we can get some free results about Hausdorff dimension. Denote the Cantor function by ϕ.

The Hausdorff Dimension of $\phi(N^*)$ *is* $\ln(2)/\ln(3)$.

This result follows straightforwardly from our previous work because the binary representation of $\phi(t)$ is obtained by replacing the 2's in the ternary representation of t by 1's. Consequently, since $3^r = 2$ and intervals of length 3^{-k} correspond to intervals of length 2^{-k} when we go from the ternary representation of t to the binary representation of $\phi(t)$, we can replace r^2 by r and $(\mathrm{ter})_d$ by $(\mathrm{bin})_d$ and modify the preceding arguments appropriately to verify that the Hausdorff Dimension of $\phi(N^*)$ is $\ln(2)/\ln(3)$.

Referring to [1–3], denote the standard Cantor set of measure $1 - \lambda$ by C_λ, $0 < \lambda < 1$; denote the corresponding Cantor functions by ϕ_λ and the corresponding nondifferentiability sets by N_λ^*.

The sets N_λ^* *and* $\phi_\lambda(N_\lambda^*)$ *have Hausdorff dimension* $\ln(2)/\ln(3)$, $0 < \lambda < 1$.

This assertion follows from the descriptions of the ϕ_λ's given in [1], Theorems 2 and 3 in [3], and the results previously established in this note. Intervals generated in the description of C_λ as an intersection of finite unions of 2^k intervals of length L_k have $L_k = (1-\lambda)2^{-k} + \lambda 3^{-k}$. The binary part of L_k overwhelms the ternary component; thus, variations of the arguments used to compute the Hausdorff dimension of $\phi(N^*)$ suffice here.

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