THE HAUSDORFF DIMENSION OF THE NONDIFFERENTIABILITY SET OF THE CANTOR FUNCTION IS $[\ln(2)/\ln(3)]^2$

RICHARD DARST

(Communicated by Andrew M. Bruckner)

Abstract. The main purpose of this note is to verify that the Hausdorff dimension of the set of points $N^*$ at which the Cantor function is not differentiable is $[\ln(2)/\ln(3)]^2$. It is also shown that the image of $N^*$ under the Cantor function has Hausdorff dimension $\ln(2)/\ln(3)$. Similar results follow for a standard class of Cantor sets of positive measure and their corresponding Cantor functions.

The Hausdorff dimension of the set of points $N^*$ at which the Cantor function is not differentiable is $[\ln(2)/\ln(3)]^2$.

Chapter 1 in [5] provides a nice introduction to Hausdorff measure and dimension; references [5-7] pursue the topic. We begin our proof with some notation and discussion. Let $C$ denote the Cantor set. Let $N^+ (N^-)$ denote the set of points at which the Cantor function does not have a right side (left side) derivative, finite or infinite. Then $N^* = N^+ \cup N^- \cup \{t : t$ is an end point of $C\}$ denotes the nondifferentiability set of the Cantor function. Although we will assume familiarity with [4], where Eidswick characterized $N^*$, some material is repeated for completeness.

A number $t$ in $C$ has a ternary representation $t = (t_1, \ldots, t_i, \ldots)$, where $t_i = 0$ or $2$.

Let $z(n)$ denote the position of the $n$th zero in the ternary representation of $t$;

1a) If $t \in N^+$, then $\limsup \{z(n+1)/z(n)\} \geq \ln(3)/\ln(2)$;

1b) $\limsup \{z(n+1)/z(n)\} > \ln(3)/\ln(2)$, then $t \in N^+$.

Let $m_d$ denote the $d$-dimensional Hausdorff measure, and put $r = \ln(2)/\ln(3)$.

We will compute the Hausdorff dimension of $N^*$ by verifying

(A) If $1 \geq d > r^2$, then $m_d N^* = 0$.

(B) If $d < r^2$, then $m_d N^* \geq K_d > 0$; $K_d$ will be specified later for a sequence of $d$'s increasing to $r^2$.

Condition (A) will be verified for each $d$ satisfying the inequalities $1 \geq d > r^2$ by constructing a set $E$ (depending on $d$) which contains $N^*$ and satisfies the equation $m_d E = 0$. To verify (B), we will consider a sequence $\{d_n\}$ of
d's increasing to \( r^2 \); for each \( d \) in the sequence, we will construct a subset \( E^* \) of \( N^* \) with \( m_d(E^*) > 0 \), which implies \( m_h(N^*) \geq m_h(E^*) = \infty \) for \( h < d \). (A) implies \( m_h(N^*) = 0 \) for \( h > r^2 \), and (B) implies \( m_h(N^*) = \infty \) for \( h < r^2 \). Consequently, the Hausdorff dimension of \( N^* \) is equal to \( r^2 \).

Verification of (A). We will use sets \( E_k = \{ t: t_k = 0 \text{ and } t_i = 2 \text{ for } k < i \leq u_k \} \), where \( u_k \) will be specified below.

Fix \( d > r^2 \). We will define a positive integer \( n \) (depending on \( d \)) and \( u_k \) for \( k \geq n \) so

\[
N^+ \subseteq \bigcup_{k \geq m} E_k, \quad m \geq n: N^+ \subseteq \limsup\{E_k\} = E^* 
\]

and

\[
2^k/(3^{u_k})^d \leq k^{-2}: k \ln(2) - du_k \ln(3) \leq -2 \ln(k) : \quad r + (2/\ln(3))(\ln(k)/k) \leq d(uk/k).
\]

The required strings of 2's in the points of \( E^* \) will be short enough to apply (1a) to verify (2), and they will be long enough to satisfy (3). Since \( d > r^2 \), put \( d = r(r + t) \), where \( t > 0 \). Then \( t = (d - r^2)/r < 1/a^* \). Choose \( n \geq 3 \) so that

\[
\ln(n)/n < t/4.
\]

Then \( \ln(m)/m \) is decreasing for \( m \geq n \) and \( 1/n < t/4 \). Thus, for \( k \geq n \) we can choose \( u_k \) so that

\[
r^{-1} - t/2 < u_k/k < r^{-1} < t/4.
\]

According to (4) and the first inequality in (5), for \( k \geq n \),

\[
r + (2/\ln(3))(\ln(k)/k) < r + 2(\ln(k)/k) < r + t/2
\]

\[
= r + t - t/2 < r + t - d(t/2) = d(r^{-1} - t/2) < d(uk/k),
\]

so (3) is satisfied for \( k \geq n \).

Referring to (1a) and the second inequality in (5), (2) is satisfied. To show that \( m_d(\limsup\{E_k\}) = m_d(E^*) = 0 \), it suffices to observe that since each \( E_i \) can be covered with \( 2^{i-1} \) intervals of length \( 3^{-u_i} \), then

\[
m_d(E^*) \leq \lim_k \sum_{i \geq k} 2^i / (3^{u_i})^d \leq \lim_k \sum_{i \geq k} i^{-2} = 0.
\]

Consequently, \( m_d N^+ = 0 \). Similarly, \( m_d N^- = 0 \); thus, \( m_d N^* = 0 \).

Verification of (B). For \( d = rsv \), where \( s = (n - 1)/n \) and \( v = rs \), we will construct a subset \( E = E_n \) of \( N^+ \) with \( m_d E \geq K = PQR \), where \( P \) and \( Q \) are positive numbers that will be defined later and \( R \) is a positive constant relating \( m_d \) and the equivalent \( d \)-dimensional net measure \( (\text{ter})_d \) obtained by requiring covers to be composed of ternary intervals \([a, b] = [i/3^k, (i + 1)/3^k]\) (which we call \( k \)-intervals) according to the inequality \( m_d \geq R(\text{ter})_d \). The existence of \( R \) follows from a variation on a theme of Besicovitch that is discussed in [5, §5.1; 7, Chapter 2, §7.1]; we are using closed ternary intervals, but only countably many end points are involved in intersections. Since \( m_d \geq R(\text{ter})_d \), we verify (B) by establishing the inequality \( (\text{ter})_d E \geq PQ \) below.

Covers are required to be ternary covers in the following discussion.
We begin by describing a generic set $E$ of the type to be used; $E$ corresponds to a sequence $0 < k_1 < u_1 < k_2 < u_2 < \cdots$ of positive integers as follows:

$$E = \{ t = (t(1), t(2), \ldots): t(k_i) = 0 \text{ and } t(k) = 2 \text{ for } k_i < k \leq u_i, \ i \geq 1 \}.$$  

The set $E$ is a closed subset of $C$ and is composed of non-end-points of $C$.

When $k_i \leq k \leq u_i$, $k$ is a fixed choice (for $E$); otherwise, $k$ is a free choice.

The strings of fixed choices will be long enough to make the points in $E$ satisfy (1b), and the strings of free choices will be long enough to assure $P > 0$ and $Q > 0$.

Let $F(p, q)$ denote the number of free choices $k$ with $p < k \leq q$.

Because of [4, Theorem 1] and the fact that

$$\liminf(3^{z(n)/2^{z(n+1)}}) \leq \liminf(3^{k_i}/2^{u_i}),$$

$E$ is a subset of $N^+$ if $\inf_i(u_i/k_i) > \ln 3/\ln 2$. In particular, recalling the definition of $u$, $E$ is a subset of $N^+$ if $u_i = v^{-1}k_i + r_i$, where $0 \leq r_i < 1$.

Let $\{(a_j, b_j)\}$ be a ternary cover of $E$. Since $E$ contains no end point of $C$, $\{(a_j, b_j)\}$ is an open cover of $E$; $E$ is compact, so we restrict attention to a finite subcover. We can also require $b_j \leq a_{j+1}$ and that $[a_j, b_j] \cap E$ be nonempty. Let $3^{-w} = \min\{b_j - a_j\}$. For $k > w$, a $k$-interval $U = [i/3^k, (i + 1)/3^k]$ intersects at most one $(a_j, b_j)$; if this intersection is nonempty, then $U \subseteq [a_j, b_j]$.

The $k_i$'s and $u_i$'s considered below are all $> w$. To prove $\text{ter}_d E \geq P Q$, it suffices to specify positive constants $P$ and $Q$ satisfying

(C) $m_d[a_j, b_j] \geq P$ (number of $u_i$-intervals in $[a_j, b_j]) 3^{-u_i}d$

(D) (number of $u_i$-intervals which intersect $E) 3^{-u_i}d \geq Q$.

Letting $[i/3^k, (i + 1)/3^k]$ denote a generic $[a_j, b_j]$, we rewrite (C) and (D) as

(C) $3^{-kd} \geq P 2^F(k, k_i)/3^{u_i}d$

(D) $2^{F(0, k_i)}/3^{u_i}d \geq Q$.

Define $u_i$ by the equation $u_i = v^{-1}k_i + r_i$, $0 \leq r_i < 1$. Thus, the points in $E$ satisfy (1b).

Define $k_{i+1}$ by specifying $F(0, k_{i+1}) = s(k_{i+1} + s_{i+1}$, where $0 \leq s_{i+1} < 1$ and $s_{i+1}$ is minimal. Such a choice is possible because for $1 < f < k$,

$$f/k - (f - 1)/(k - 1) = (k - f)/[k(k - 1)] < k^{-1}.$$  

This definition of $k_{i+1}$ provides enough free choices to assure $P > 0$ and $Q > 0$.

Verification of (C).

$$3^{-kd} \geq P 2^F(k, k_i)/3^{u_i}d \Leftrightarrow 3^{(u_i-k)d} \geq P 2^F(k, k_i) \Leftrightarrow 2^{u_i-k}sv P 2^F(k, k_i) \Leftrightarrow 1 \geq P 2^{F(k, k_i) - vs[u_i-k]}.$$  

Put $h(k, i) = F(k, k_i) - sv(u_i - k)$. If $k_j \leq k \leq u_j$, then $h(k, i) \leq h(u_j, i)$; and if $u_{j-1} < k < k_j$, then $h(k, i) < h(u_{j-1}, i)$. Thus, it suffices to show that $h(u_j, i)$ is bounded for $j < i$.

$$F(u_j, k_i) - sv(u_i - u_j) = [(sk_i + s_i) - (sk_j + s_j)] - sv[(v^{-1}k_i + r_i) - (v^{-1}k_j + r_j)]$$

$$= [s_i - s_j] - sv[r_i - r_j] < 1 + sv.$$  

Hence, we put $P = 2^{-1+sv}$.
Verification of (D). \( F(0, k_i) = sk_i + s_i, u_i d = (v^{-1}k_i + r_i)rsv = rs(k_i + vr_i), \) and \( 3' = 2; \) consequently, \( 2F(0, k_i)/3u_i d = 2(s_i - svr_i) \geq 2^{-sv}. \) Thus, putting \( Q = 2^{-sv}, \) we have shown that the Hausdorff dimension of \( N^* \) is \( r^2. \)

Now we can get some free results about Hausdorff dimension. Denote the Cantor function by \( \phi. \)

The Hausdorff Dimension of \( \phi(N^*) \) is \( \ln(2)/\ln(3). \)

This result follows straightforwardly from our previous work because the binary representation of \( \phi(t) \) is obtained by replacing the 2's in the ternary representation of \( t \) by 1's. Consequently, since \( 3' = 2 \) and intervals of length \( 3^{-k} \) correspond to intervals of length \( 2^{-k} \) when we go from the ternary representation of \( t \) to the binary representation of \( \phi(t), \) we can replace \( r^2 \) by \( r \) and \( (\text{ter})_d \) by \( (\text{bin})_d \) and modify the preceding arguments appropriately to verify that the Hausdorff Dimension of \( \phi(N^*) \) is \( \ln(2)/\ln(3). \)

Referring to [1–3], denote the standard Cantor set of measure \( 1 - \lambda \) by \( C_\lambda, 0 < \lambda < 1; \) denote the corresponding Cantor functions by \( \phi_\lambda \) and the corresponding nondifferentiability sets by \( N_\lambda^*. \)

The sets \( N_\lambda^* \) and \( \phi_\lambda(N_\lambda^*) \) have Hausdorff dimension \( \ln(2)/\ln(3), \) \( 0 < \lambda < 1. \)

This assertion follows from the descriptions of the \( \phi_\lambda \)'s given in [1], Theorems 2 and 3 in [3], and the results previously established in this note. Intervals generated in the description of \( C_\lambda \) as an intersection of finite unions of \( 2^k \) intervals of length \( L_k \) have \( L_k = (1 - \lambda)2^{-k} + \lambda 3^{-k}. \) The binary part of \( L_k \) overwhelms the ternary component; thus, variations of the arguments used to compute the Hausdorff dimension of \( \phi(N^*) \) suffice here.

References


Department of Mathematics, Colorado State University, Fort Collins, Colorado 80523