ALMOST ISOMETRIC COPIES OF $l_\infty$
IN SOME BANACH SPACES

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(Communicated by Palle E. T. Jorgensen)

Abstract. It is shown that any $\sigma$-complete Banach lattice, with an order semi-
continuous norm containing an isomorphic copy of $l_\infty$, contains an almost
isometric copy of $l_\infty$. It is also proved that any Fenchel-Orlicz space (resp. the
subspace of finite elements of any Fenchel-Orlicz space) generated by an Orlicz
function not satisfying the suitable $\Delta_2$-condition contains an almost isometric
copy of $l_\infty$ (resp. $c_0$).

1. Introduction

Two Banach spaces $X$, $Y$ are said to be $(1 + \varepsilon)$-isometric provided there
exists a linear isomorphism $T : X \to Y$ such that $\|T\|\|T^{-1}\| \leq 1 + \varepsilon$. By
scaling we can arrange that $T$ is a $(1 + \varepsilon)$-isometry if

$$\|x\|_X \leq \|Tx\|_Y \leq (1 + \varepsilon)\|x\|_X$$

for any $x \in X$. We say that a Banach space $X$ contains an almost isometric
copy of $Y$ if for any $\varepsilon > 0$ there exists a subspace $Z$ in $X$ such that $Z$, $Y$
are $(1 + \varepsilon)$-isometric.

Note that Krivine [Kr] proved that if a Banach space $X$ contains $l_p$’s $(1 + \varepsilon)$-
uniformly for some $\varepsilon > 0$, $1 \leq p \leq \infty$, then it also contains them almost
isometrically. For $p = 1$ or $p = \infty$, this result goes back to James and for
$p = 2$ to Dvoretzky. Let us recall that the well-known result of James [J]
shows that a Banach space $X$ contains an almost isometric copy of $c_0$ (or $l_1$)
whenever it contains an isomorphic copy of $c_0$ (resp. $l_1$).

In this paper we show a similar result for $l_\infty$-copies; however, we restrict
ourselves only to special Banach spaces.

Note also that, as far as we know, for $1 < p < \infty$ it is unknown whether or
not any Banach space isomorphic to $l_p$ contains a subspace almost isometric to
$l_p$. This is known as the “distortion problem” [LP].

In the sequel $X$ denotes a Banach space; $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{R}_+^\infty$ stand for the
reals, nonnegative reals, and extended (by $+\infty$) nonnegative reals. In what
follows $(\Omega, \Sigma, \mu)$ denotes an arbitrary $\sigma$-finite measure space. For the sake
of simplicity we will consider only nonatomic and purely atomic (the counting)
measure. $L^0(\mu, X)$ denotes the space of all (equivalence classes of) strongly \(\Sigma\)-measurable functions defined on \(\Omega\) with values in \(X\).

A map \(\Phi : X \to \mathbb{R}_+\) is said to be an Orlicz function if \(\Phi(0) = 0\), \(\Phi\) is continuous at 0, lower semicontinuous on \(X\), even, and

\[(*) \quad \Phi \text{ is bounded on any ball in } X, \]
\[(**) \quad \inf \{\Phi(x) : \|x\| = r\} \to \infty \text{ as } r \to \infty.\]

Given an arbitrary Orlicz function \(\Phi\) we define a functional \(I_\Phi : L^0(\mu, X) \to \mathbb{R}_+\) by

\[I_\Phi(f) = \int_\Omega \Phi(f(t)) \, d\mu,\]

which is even and convex, \(I_\Phi(0) = 0\), and for any \(f \in L^0(\mu, X)\) the condition \(I_\Phi(\lambda f) = 0\) for any \(\lambda > 0\) yields \(f = 0\).

The Fenchel-Orlicz space \(L^\Phi(\mu, X)\) generated by an Orlicz function \(\Phi\) is defined as the set of all functions \(f \in L^0(\mu, X)\) such that \(I_\Phi(\lambda f) < \infty\) for some \(\lambda > 0\) depending on \(f\) (cf. [T2] and in the scalar case also [KR, Lu, M]).

The subspace \(E^\Phi(\mu, X)\) of \(L^\Phi(\mu, X)\) (called the subspace of finite elements) is defined by

\[E^\Phi(\mu, X) = \{f \in L^\Phi(\mu, X) : I_\Phi(\lambda f) < \infty \text{ for any } \lambda > 0\}\]

The spaces \(L^\Phi\) and \(E^\Phi\) can be equipped with the Luxemburg norm

\[\|f\| = \inf \{\varepsilon > 0 : I_\Phi(f/\varepsilon) \leq 1\}\]

as well as with the Orlicz norm

\[\|f\|_0 = \sup \left\{ \int_\Omega \langle f(t), g(t) \rangle \, d\mu : g \in L^0(\mu, X^*), \ I_\Phi^*(f) \leq 1 \right\},\]

where \(X^*\) denotes the dual space of \(X\) and \(\Phi^*\) is the complementary function to \(\Phi\) in the sense of Young, i.e.,

\[\Phi^*(x^*) = \sup \{\langle x, x^* \rangle - \Phi(x) : x \in X\}\]

for any \(x^* \in X^*\). It is well known (cf. [N] and in the scalar case also [KR, RR]) that

\[\|f\|_0 = \inf \{k^{-1}(1 + I_\Phi(kf)) : k > 0\} .\]

Furthermore, \((L^\Phi(\mu, X), \|\cdot\|)\) is a Banach space (cf. [T2] and in the scalar case also [KR, Lu, M]).

We say that an Orlicz function \(\Phi\) satisfies the \(\Delta_2\)-condition for all \(x \in X\) (at infinity) [at zero] if there are positive constants \(K\) and \(C\) such that \(\Phi(2x) \leq K\Phi(x)\) for all \(x \in X\) (for \(x \in X\) satisfying \(\Phi(x) \geq c\) [for \(x \in X\) satisfying \(\Phi(x) \leq c\)].

An Orlicz function \(\Phi\) satisfies the suitable \(\Delta_2\)-condition if it satisfies the \(\Delta_2\)-condition for all \(x \in X\) when \(\mu\) is nonatomic infinite, the \(\Delta_2\)-condition at infinity if \(\mu\) is nonatomic finite, and the \(\Delta_2\)-condition at zero if \(\mu\) is the counting measure (cf. [RR]).

It is known (cf. [H] and in the scalar case also [K, T1, T2]) that any Fenchel-Orlicz space \(L^\Phi(\mu, X)\), with the Luxemburg norm, containing an isomorphic copy of \(l_\infty\) contains also an isometric copy of \(l_\infty\).
However, the Orlicz space \( (L^\Phi(\mu, \mathbb{R}), \| \cdot \|) \) need not contain an isometric copy of \( l_\infty \) whenever it contains an isomorphic copy of \( l_\infty \) (this follows from the criteria for rotundities of \( L^\Phi(\mu, \mathbb{R}) \) equipped with the Orlicz norm (cf. [T1, RR]). It will be proved in this paper that in the case of the Orlicz norm the Fenchel-Orlicz space \( L^\Phi(\mu, X) \) contains an almost isometric copy of \( l_\infty \) whenever it contains an isomorphic copy of \( l_\infty \).

Recall that a Banach lattice \( X \) is said to be \( \sigma\)-complete if every order bounded sequence in \( X \) has a supremum. A \( \sigma\)-complete Banach lattice \( X \) is said to have an order continuous norm (an order semicontinuous norm) whenever \( x_n \downarrow 0 \) implies \( \|x_n\| \to 0 \) (resp. \( 0 \leq x_n \uparrow x, x \in X \), implies \( \|x_n\| \to \|x\| \)).

2. Results

We start with the following

**Theorem 1.** Let \( E \) be a \( \sigma\)-complete Banach lattice with an order semicontinuous norm. If \( E \) contains an isomorphic copy of \( l_\infty \), then it contains an almost isometric copy of \( l_\infty \).

**Proof.** From the well-known result of Lozanovskii [L] it follows that the norm in \( E \) is not order continuous whenever \( E \) contains an isomorphic copy of \( l_\infty \). Since \( E \) is \( \sigma\)-complete (by the assumption), in virtue of the result of Ando [A] (cf. also [KA, p. 382; LT, p. 7]) it follows that there exists an order bounded sequence \( (u_k) \) of mutually disjoint positive elements in \( E \) satisfying \( c = \inf_n \|u_n\| > 0 \).

Assume that \( 0 \leq u_n \leq x \in X \) holds for all \( n \in \mathbb{N} \) and put

\[
k_n = \sup \left\{ \left\| \sum_{i=n}^m u_i \right\| : m \in \mathbb{N} \right\}
\]

for \( n \in \mathbb{N} \). Since \( (K_n) \) is a nonincreasing sequence satisfying \( c \leq K_n \leq \|x\| \), it is convergent and \( c \leq K = \lim_{n \to \infty} K_n \leq \|x\| \).

Let \( 0 < \varepsilon < 1 \) be fixed. Take \( 0 < \theta < 1 < \eta \) such that \( \theta/\eta > 1 - \varepsilon \). Now pick \( k_1 \) in such a way that \( K_{k_1} < \eta K \). It follows from the definition of \( K_n \) that for a certain \( k_2 > k_1 \), we get

\[
\left\| \sum_{i=k_1}^{k_2-1} u_i \right\| > \theta K_{k_1} \geq \theta K.
\]

We can construct, by the induction, an increasing sequence \( (k_n) \) in \( \mathbb{N} \) such that

\[
\left\| \sum_{i=k_{n+1}}^{k_{n+1}-1} u_i \right\| > \theta K,
\]

\[
K_{k_n} < \eta K.
\]

Now consider the sequence \( (x_n) \) of positive and pairwise disjoint elements, where

\[
x_n = \sum_{i=k_n}^{k_{n+1}-1} (\eta K)^{-1} u_i.
\]
For each $0 \leq \zeta = (\zeta_n) \in l_\infty$, we put
\[ T(\zeta) = \sup \left\{ \sum_{i=1}^n \xi_i x_i : n \in \mathbb{N} \right\}. \]

The supremum exists because $E$ is $\sigma$-order continuous and
\[ \sum_{i=1}^n \xi_i x_i \leq (\eta K)^{-1} x \|\xi\|_\infty. \]

Since $T : l_\infty^+ \to E^+$ and $T$ is linear, it extends uniquely to a linear operator $T$ from $l_\infty$ into $E$.

If $\xi = (\zeta_n) \in l_\infty$, then the order semicontinuity of the norm and (2) yield
\[ \|T(\xi)\| \leq \|T(|\xi|)\| = \|T(|\xi|)\| \]
\[ = \lim_{n \to \infty} \left( \sum_{i=1}^n |\xi_i| x_i \right) \leq \|\xi\|_\infty \sup_{n} \left( \sum_{i=1}^n x_i \right) \]
\[ \leq (\eta K)^{-1} K_i \|\xi\|_\infty \leq \|\xi\|_\infty. \]

Furthermore, by (1)
\[ \|T \xi\| \geq |\zeta_n| \|x_n\| \geq \theta \eta^{-1} |\zeta_n| \geq (1 - \varepsilon)|\zeta_n| \]
for all $n \in \mathbb{N}$. Consequently,
\[ (1 - \varepsilon)|\zeta\|_\infty \leq \|T \xi\| \leq \|\xi\|_\infty \]
for any $\xi \in l_\infty$. This finishes the proof.

Note that Fenchel-Orlicz spaces are not isomorphic to Banach lattices in general. Because of this we are interested in the problem of almost isometric copies of $l_\infty$ as well as $c_0$ in these spaces. In order to present our results we first need to prove some auxiliary lemmas.

**Lemma 1.** If an Orlicz function $\Phi$ does not satisfy the suitable $A_2$-condition, then there exists a sequence $(f_n)$ of functions with pairwise disjoint supports and such that $\|f_n\|_\Phi = 1$ for any $n \in \mathbb{N}$ and $I_\Phi(f_n) \to 0$ as $n \to \infty$.

**Proof.** It is known (cf. [H, K, T1]) that if $\Phi$ does not satisfy the suitable $A_2$-condition, then there exists a sequence $(g_n)$ in $L^\Phi(\mu, X)$ with pairwise disjoint supports and such that $I_\Phi(g_n) \to 0$ and $\|g_n\| = 1$ for any $n \in \mathbb{N}$. We have $\|g_n\|_0 \geq 1$, whence defining $f_n = g_n/\|g_n\|_0$ we obtain $\|f_n\|_0 = 1$ and $0 \leq I_\Phi(f_n) \leq I_\Phi(g_n) \to 0$, i.e., $I_\Phi(f_n) \to 0$.

**Lemma 2.** For any $f \in L^\Phi(\mu, X)$ and $\delta > 0$ we have $\|f\|_0 \leq 1 + \delta$ whenever $I_\Phi(f) \leq \delta$.

**Proof.** Under the assumptions we have
\[ \|f\|_0 = \inf_{k > 0} k^{-1}(1 + I_\Phi(k f)) \leq 1 + I_\Phi(f) \leq 1 + \delta. \]

**Theorem 2.** If $\Phi$ is an Orlicz function on $X$ not satisfying the suitable $A_2$-condition, then for any $\varepsilon > 0$ there exists a $(1 + \varepsilon)$-isometry $T$ of $l_\infty$ into $(L^\Phi(\mu, X), \|\cdot\|_0)$ such that $T c_0 \not\subset E^\Phi(\mu, X)$.

**Proof.** Take an arbitrary $\varepsilon > 0$. Let $(f_n)$ be the sequence in $L^\Phi(\mu, X)$ with pairwise disjoint supports such that
\[ I_\Phi(f_n) \leq 2^{-n} \varepsilon \quad \text{and} \quad \|f_n\|_0 = 1 \]
for any $n \in \mathbb{N}$, which is built as in the proof of Lemma 1. Define an operator $T : l_\infty \to L^\Phi(\mu, X)$ by $T\xi = \sum_{n=1}^{\infty} \xi_n f_n$ for any $\xi = (\xi_n) \in l_\infty$. We have by the orthogonal additivity of $I_\Phi$

$$I_\Phi(T\xi/\|\xi\|_\infty) = \sum_{n=1}^{\infty} I_\Phi(\xi_n f_n/\|\xi\|_\infty) \leq \sum_{n=1}^{\infty} I_\Phi(f_n) \leq \varepsilon.$$ 

Thus, from Lemma 2 it follows that

$$\|T\xi\|_0 \leq (1 + \varepsilon)\|\xi\|_\infty$$

for every $\xi \in l_\infty$. Since $\|T\xi\|_0 \geq \|\xi_n f_n\|_0 = |\xi_n|$ for any $n \in \mathbb{N}$, we have

$$\|T\xi\|_0 \geq \|\xi\|_\infty.$$ 

This means that $T$ is an $(1 + \varepsilon)$-isometry. Note that $Te_n = f_n$, where $e_n$ denotes the sequence of real numbers whose $n$th term is one and the rest are zero. We have $f_n = g_n/\|g_n\|^0$, where $I_\Phi(g_n) \to 0$ as $n \to \infty$ and $\|g_n\| = 1$ for $n \in \mathbb{N}$. Hence $I_\Phi(\lambda g_n) = \infty$ for any $\lambda > 1$ and $n \in \mathbb{N}$ large enough. Thus, it follows that $f_n \notin E_\Phi(\mu, X)$ for $n \in \mathbb{N}$ large enough. This finishes the proof.

Theorem 3. If $\Phi$ is an Orlicz function on $X$ not satisfying the suitable $\Delta_2$-condition, then for any $\varepsilon > 0$ there exists an operator $T : l_\infty \to L^\Phi(\mu, X)$, which is a $(1 + \varepsilon)$-isometry in the case of the Luxemburg norm as well as in the case of the Orlicz norm. Furthermore, $T$ restricted to $c_0$ acts $(1 + \varepsilon)$-isometrically to $E(\mu, X)$ with respect to both norms in $E(\mu, X)$.

Proof. Note that $\Delta_2$-condition is equivalent to the $\Delta_l$-condition for any $l > 1$. Note also that in view of conditions ($\ast$) and ($\ast\ast$) an Orlicz function $\Phi$ satisfies the suitable $\Delta_2$-condition for some constant $c > 0$ if and only if it satisfies this condition for an arbitrary constant $c > 0$. Therefore, if $\Phi$ does not satisfy the $\Delta_2$-condition at infinity, then for any $\varepsilon > 0$ we can choose a sequence $(x_n)$ in $X$ and a sequence $(A_n)$ in $\Sigma$ such that

\begin{align*}
(3) \quad & \Phi((1 + \varepsilon)x_n) > 2^{n+1}\varepsilon^{-1}\Phi(x_n), \\
(4) \quad & 2^{-n-1}\varepsilon < \Phi(x_n)\mu(A_n) \leq 2^{-n}\varepsilon
\end{align*}

for any $n \in \mathbb{N}$. Note that in the case of a nonatomic measure we can get (4) in the form $\Phi(x_n)\mu(A_n) = 2^{-n}\varepsilon$.

Now, let $T$ be an operator defined by

$$T\xi = \sum_{n=1}^{\infty} \xi_n x_n \chi_{A_n}$$

for any $\xi = (\xi_n) \in l_\infty$.

If $\xi = (\xi_n) \in l_\infty$, then

$$I_\Phi(T\xi/\|\xi\|_\infty) = \sum_{n=1}^{\infty} \Phi(x_n)\mu(A_n) \leq \varepsilon,$$

i.e., $T\xi \in L^\Phi(\mu, X)$.
Taking any \( \xi = (\xi_n) \in c_0 \) and any \( \lambda > 0 \) we can choose \( m \in \mathbb{N} \) such that \( \lambda |\xi_n| \leq 1 \) for \( n \geq m \). Hence
\[
I_\Phi(\lambda T\xi) = \sum_{n=1}^{m-1} \Phi(\lambda \xi_n x_n) \mu(A_n) + \sum_{n=m}^{\infty} \Phi(\lambda \xi_n x_n) \mu(A_n)
\leq \sum_{n=1}^{m-1} \Phi(\lambda \xi_n x_n) \mu(A_n) + \sum_{n=m}^{\infty} \Phi(x_n) \mu(A_n)
\leq \sum_{n=1}^{m-1} \Phi(\lambda \xi_n x_n) \mu(A_n) + \varepsilon \sum_{n=m}^{\infty} 2^{-n} < \infty,
\]
i.e., \( T\xi \in E^\Phi(\mu, X) \) for any \( \xi \in c_0 \).

Moreover, the right inequality in (4) yields
\[
\|T\xi\| \leq \inf_{k > 0} k^{-1}(1 + I_\Phi(k T\xi/\|\xi\|_\infty))
\leq 1 + I_\Phi(T\xi/\|\xi\|_\infty) \leq 1 + \varepsilon,
\]
whence
\[
\|T\xi\| \leq (1 + \varepsilon) \|\xi\|_\infty
\]
for any \( \xi \in l_\infty \).

On the other hand, for \( 0 \leq \varepsilon < 1 \) and some \( n \in \mathbb{N} \), applying (3) and the left inequality in (4), we get
\[
I_\Phi((1 + 2\varepsilon) T\xi/\|\xi\|_\infty) \geq \Phi((1 + \varepsilon) x_n) \mu(A_n)
> 2^{n+1} \varepsilon^{-1} \Phi(x_n) \mu(A_n) > 1,
\]
which yields
\[
\|T\xi\|^0 \geq \|T\xi\| \geq (1 + 2\varepsilon)^{-1} \|\xi\|_\infty
\]
for any \( \xi \in l_\infty \). This finishes the proof.

\textit{Added in Proof.} Recently, E. Odell and T. Schlumprecht have answered the distortion problem negatively.

\textbf{Acknowledgment}

The authors thank the referee for valuable remarks.

\textbf{References}


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