PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 119, Number 1, September 1993

INJECTIVE MORPHISMS OF AFFINE VARIETIES

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(Communicated by Louis J. Ratliff, Jr.)

ABSTRACT. In this note an elementary proof that every injective morphism from an affine variety into itself is necessarily surjective is given.

1. INTRODUCTION

Let K be any algebraically closed field and V an algebraic variety defined over K. It is known that any injective morphism from V into itself is necessarily surjective [7, Proposition (10.4.11), p. 103; 1; 2; 6]. Borel remarked that Shimura had a proof of this theorem by reduction modulo p [6]. It seems that such a proof has not been published yet. The purpose of this note is to provide an elementary proof along this line when V is an affine variety. What we shall prove is the following

Theorem. *Let K be any algebraically closed field, V an affine variety defined over K, and $\varphi\colon V \to V$ a morphism from V into itself. If φ is injective, then it is surjective.*

By [2, p. 3] the general case when V is any algebraic variety follows from the above theorem. In fact, we shall present a proof for the general case in the appendix using Shimura's reduction theory, although we do not know whether this proof is what Shimura had in mind. For additional information when V is an affine space see [10; 5; 3, (2.1) Theorem].

2. THE PROOF OF THE THEOREM

The essence of our proof goes back to an idea of Shafarevich about p-group actions on affine spaces [4, Lemma; 8, Theorem 4.1].

Let V be an affine variety in $\mathbf{A}^n$, the affine n-space. Denote the polynomial ring of n variables over K by $K[X_1, \dots, X_n]$. Let I be the defining ideal of V and $g_1, g_2, \dots, g_s$ a set of generators of I. Denote the coordinate ring of V by

$$R := K[X_1, \dots, X_n]/I = K[x_1, \dots, x_n],$$

Received by the editors October 30, 1991 and, in revised form, January 8, 1992.
1991 *Mathematics Subject Classification.* Primary 14A10; Secondary 13B10.
Key words and phrases. Injective morphisms of affine varieties.
This work was done while the author was visiting Purdue University and was partially supported by the National Science Council of Republic of China.

0002-9939/93 \$1.00 + \$.25 per page

where $g(X)$ is regarded as an element in $K[X_1, \ldots, X_n]$, while $g(x)$ is regarded as an element in R. Points in V or $\mathbf{A}^n$ will be denoted by $(a_1, \ldots, a_n)$, $(b_1, \ldots, b_n)$ or simply by a, b.

Let ϕ be a morphism from V into V given by

$$\begin{aligned}\phi\colon V &\to V\\ a &\mapsto (f_1(a), f_2(a), \ldots, f_n(a))\end{aligned}$$

where $f_1(x), \ldots, f_n(x) \in R$. Let $V \times V$ be the product space of V with itself, and consider the morphism

$$\begin{aligned}\Phi\colon V \times V &\to \mathbf{A}^n\\ (a, b) &\mapsto (f_1(a) - f_1(b), f_2(a) - f_2(b), \ldots, f_n(a) - f_n(b)).\end{aligned}$$

Denote the diagonal of $V \times V$ by

$$\Delta := \{(a_1, \ldots, a_n, a_1, \ldots, a_n) \in V \times V : a_i \in K\}.$$

It is clear that

$$\begin{aligned}&\phi \text{ is injective}\\ &\Leftrightarrow \Phi^{-1}(0, \ldots, 0) = \Delta.\\ &\Leftrightarrow \text{there exists an integer } m \text{ such that}\end{aligned}$$

$$(x_i - y_i)^m = \sum_{j=1}^{n} h_{ij}(x, y)\{f_j(x) - f_j(y)\} \quad \text{for } 1 \le i \le n \tag{1}$$

for some $h_{ij}(x, y) \in R \otimes_K R = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$, where we identify x_i and y_j with $x_i \otimes 1$ and $1 \otimes y_j$ in $R \otimes_K R$.

Suppose that ϕ is not surjective and $c = (c_1, \ldots, c_n)$ is not in $\phi(V)$. Then the system of equations

$$f_1(x) = c_1, \ldots, f_n(x) = c_n$$

has no solution in V. Therefore, by Hilbert's Nullstellensatz, there exist $h_1(x), \ldots, h_n(x) \in R$ so that

$$\sum_{i=1}^{n} h_i(x)\{f_i(x) - c_i\} = 1. \tag{2}$$

Collect all the coefficients of $g_1(X), \ldots, g_s(X), f_1(X), \ldots, f_n(X), h_{ij}(X, Y), h_i(X)$, and all the $c_1, c_2, \ldots, c_n$. Call this set $\{d_1, d_2, \ldots, d_t\}$. Define a subring S of K by

$$S := \begin{cases} Z[d_1, d_2, \ldots, d_t] & \text{if } \operatorname{char} K = 0,\\ Z_p[d_1, d_2, \ldots, d_t] & \text{if } \operatorname{char} K = p > 0.\end{cases}$$

By Nagata's version of Noether's normalization lemma, there is a maximal ideal M of S so that $k := S/M$ is a finite field [9].

Let W be the affine n-space over k and W_0 the affine variety in W defined by

$$W_0 = \{a \in W : \overline{g}_1(a) = \cdots = \overline{g}_s(a) = 0\},$$

where $\overline{g}$ is the image of g when passing from S onto k.

Thus we obtain an injective morphism ϕ_0 given by

$$\begin{aligned}\phi_0\colon W_0 &\to W_0\\ a &\mapsto (\overline{f}_1(a), \dots, \overline{f}_n(a)),\end{aligned}$$

which is not surjective because the image of $c = (c_1, \dots, c_n)$ is in W_0 and both formulae (1) and (2) still hold when passing to k. Remember, both W and W_0 are finite sets. Thus we get a one-to-one but not onto map from a nonempty finite set to itself—a contradiction.

APPENDIX

In this appendix we shall establish the theorem when the variety is any quasi-projective variety by applying Shimura's theory of reduction modulo p of algebraic varieties [11; 12, Appendix; 13, §9].

We recall some fundamental facts of Shimura's reduction theory. Let K and K' be two fixed universal domains. We only deal with specializations defined on a subfield of K taking values in K'. Suppose that k and k' are subfields of K and K', respectively, $\lambda\colon k \to k'$ is a specialization from k onto k', and V is a quasi-projective algebraic variety defined over k. Then V has a unique specialization over λ, which we denote by $\overline{V}$. The notion of λ-simple varieties is introduced in [11, p. 163; 13, p. 83]. For a λ-simple quasi-projective variety V, the specialization of V over λ preserves inclusion, sum, intersection-product, direct product, and projection [11, Theorems 17, 18, 19; 13, Proposition 1]. Moreover, if V is irreducible and x is a generic point over k, then, as a point set, $\overline{V}$ is equal to the set of all specializations of x over λ (into the universal domain K') [12, Lemma 3].

Now we may start to prove

Theorem A. *Let K be a universal domain, V any quasi-projective variety over K, and $\varphi\colon V \to V$ a morphism from V into itself. If φ is injective, then it is surjective.*

Proof. As in the proof of §2, define a subring S of K in a similar way so that both V and φ are defined over k_0 and V has a k_0-rational point, where k_0 is the quotient field of S. By [12, Lemma 6] adjoin a finite number of nonzero elements of k_0 and their inverses to S so as to assure λ-simplicity. By abuse of language we still denote by S this enlarged finitely generated ring.

Again by Noether's normalization lemma, find a homomorphism $\lambda\colon S \to k_0'$, where k_0' is some finite field. Extend λ to a specialization of k_0; we still call it λ. Let K' be a fixed universal domain containing k_0'. Then $\overline{V}$, the specialization of V over λ, is defined over k_0' and is nonempty and λ-simple.

Let Γ be the graph of $\varphi\colon V \to V$. Then $\overline{\Gamma}$, the specialization of Γ, is the graph of the endomorphism $\overline{\varphi}\colon \overline{V} \to \overline{V}$. Note that $\overline{\varphi}$ is injective again. For if ξ is any point of $\overline{V}$ over K', choose a point x of V so that ξ is a specialization of x by [12, Lemma 3]. Then

$$(\overline{V} \times \{\xi\}) \cdot \overline{\Gamma} = \overline{(V \times \{x\}) \cdot \Gamma} \tag{3}$$

is either empty or consists of one point only. Assume that we have established the surjectivity of $\overline{\varphi}$. Then, again by (3), we find that $(V \times \{x\}) \cdot \Gamma$ is not empty; therefore, φ is onto.

To prove the surjectivity of $\overline{\varphi}$, it suffices to prove the surjectivity of $\overline{\varphi}|_{\overline{V}(\overline{k_0'})}$, where $\overline{V}(\overline{k_0'})$ is the set of points on $\overline{V}$, whose coordinates are algebraic over k_0'. Now let k_1' be any finite extension field of k_0' and $\overline{V}(k_1')$ the set of points on $\overline{V}$, whose coordinates are in k_1'. Since $\overline{\varphi}\colon \overline{V}(k_1') \to \overline{V}(k_1')$ is an injective map of a nonempty finite set into itself, it is onto. Hence $\varphi|_{\overline{V}(\overline{k_0'})}$ is surjective.

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