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UNIVALENT HARMONIC MAPPINGS ON $\Delta = \{z: |z| > 1\}$

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ABSTRACT. The purpose of this paper is to study univalent harmonic functions defined in $\Delta = \{z: |z| > 1\}$ from the point of view of function theory. Estimates are given for all Fourier coefficients in a normalized class of mappings. Some explicit mappings are examined because of their extremal character.

1. INTRODUCTION

Recently, Hengartner and Schober [3] studied the class Σ_H of all complex-valued, harmonic, orientation-preserving, univalent mappings f defined on $\Delta = \{z: |z| > 1\}$, which are normalized at infinity by $f(\infty) = \infty$. Such functions admit the representation

$$f(z) = h(z) + \overline{g(z)} + A \log |z| \tag{1.1}$$

where

$$h(z) = \alpha z + \sum_{k=0}^{\infty} a_k z^{-k} \quad \text{and} \quad g(z) = \beta z + \sum_{k=1}^{\infty} b_k z^{-k}$$

are analytic in Δ and $0 \le |\beta| < |\alpha|$. In addition, $\tilde{a} = \overline{f_{\bar{z}}}/f_z$ is analytic and satisfies $|\tilde{a}(z)| < 1$. Also one can easily show that $|A|/2 \le |\alpha| + |\beta|$ using the bound $|s_1| \le 1 - |s_0|^2$ for analytic functions $\tilde{a}(z) = s_0 + s_1 z^{-1} + \cdots$ in Δ that are bounded by one.

Our purpose is to continue the investigation of this class Σ_H. In §2 we discuss the Poisson integral representation and continuous extension to the boundary of a harmonic mapping of Δ onto a domain whose complement is a strictly convex set. We obtain the sharp bounds for the Fourier coefficients (1.1) using the length of $f(|z| = 1)$ when the harmonic function $f \in \Sigma_H$ is of bounded variation on $|z| = 1$. Specializing to the case where the target region is also the exterior of the unit disk, we obtain the estimates

$$|\alpha + \overline{b}_1| \le 1, \qquad |\beta + \overline{a}_1| \le 1;$$
$$|b_n| \le 1/n, \quad |a_n| \le 1/n \qquad \text{for } n \ge 2$$

for the Fourier coefficients in (1.1).

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In §3 we show that the bounds for b_n $(n \geq 2)$ obtained in §2 are sharp by finding extremal functions when the target region is the exterior of the unit disk.

2. The Poisson integral representation

Let $f(\theta)$ be integrable on $[0, 2\pi]$ and

$$u(z) = u(re^{i\theta}) = -\frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\left[\frac{e^{i\varphi}+z}{e^{i\varphi}-z}\right] f(\varphi)\,d\varphi \qquad (r > 1)$$

be the Poisson integral of $f(\theta)$. Then $u(z)$ is harmonic in $|z| > 1$.

Lemma 2.1. (a) *If $f(\theta)$ is continuous at θ_0, then $u(z) \to f(\theta_0)$, as $z = re^{i\theta}$ approaches the point $e^{i\theta_0}$ by any mode of approach through points in $|z| > 1$.*

(b) *Suppose that $f(\theta)$ is discontinuous at θ_0, such that $f(\theta_0+0)$, $f(\theta_0-0)$ exist. Let $l_\psi(e^{i\theta})$ be a segment in $|z| \leq 1$ with endpoint $e^{i\theta}$, making an angle ψ $(0 < \psi < \pi)$ with the positive tangent of $|z| = 1$ at $e^{i\theta}$. If $z \to e^{i\theta_0}$ along $L_\psi(e^{i\theta_0}) = \{z\colon 1/\overline{z} \in l_\psi(e^{i\theta_0})\}$ or any path tangent to $L_\psi(e^{i\theta_0})$ at $e^{i\theta_0}$, then $u(z) \to f(\theta_0+0) + (\psi/\pi)[f(\theta_0-0) - f(\theta_0+0)]$.*

Proof. These follow from the interior version of Theorems IV.2 and IV.3 in [5] by replacing z by $1/\overline{z}$.

Recall that the radial limits of a bounded harmonic function in the open unit disk exist almost everywhere on $|z| = 1$.

Theorem 2.2. *Suppose that f is a complex-valued, harmonic, orientation-preserving, univalent mapping from $\Delta = \{z\colon |z| > 1\}$ onto the exterior U of a strictly convex Jordan curve Γ with $f(\infty) = \infty$. Then f has a continuous extension to $\overline{\Delta}$.*

Proof. Since $f \in \Sigma_H$, f has the representation (1.1). Consider $f(z) - \alpha z - \overline{\beta z} - A\log|z|$. Then $f(1/z) - \alpha/z - \overline{\beta/z} + A\log|z|$ is a bounded harmonic function in $|z| < 1$. Thus, $\operatorname{Lim}_{r\to 1^-}[f(r^{-1}e^{-i\theta}) - \alpha r^{-1}e^{-i\theta} - \overline{\beta} r^{-1}e^{i\theta} + A\log r]$ exists a.e. Therefore $\operatorname{Lim}_{r\to 1^+} f(re^{i\theta})$ exists a.e. and belongs to Γ.

A univalent analytic mapping ϕ from Δ onto U extends homeomorphically to $\overline{\Delta}$, and $\phi(\partial\Delta) = \Gamma$ (cf. [4, Theorem on correspondence of boundaries, p. 14]). Let $\tilde{\psi} = \phi^{-1}$ in U. Then $\tilde{\psi}\circ f$ is an orientation-preserving homeomorphism of Δ onto itself. Its radial limit function $\tilde{h}$ exists and has modulus 1 a.e. on $\partial\Delta$. By redefining $\tilde{h}$ on a set of measure zero, we may write $\tilde{h}(e^{i\theta}) = e^{i\eta(\theta)}$, where η is a nondecreasing function on $\mathbb{R}$ and $\eta(\theta + 2\pi) = \eta(\theta) + 2\pi$.

Define E_0 to be the at most countable set of points $e^{i\theta}$ on $\partial\Delta$ that correspond to the discontinuities, which are finite jumps, of η. On $(\partial\Delta)\backslash E_0$ the function $\phi\circ\tilde{h}$ is continuous and its values belong to Γ. At the points of the countable set E_0, the function $\phi\circ\tilde{h}$ has one-sided limits, which also belong to Γ since Γ is closed.

Now $\phi\circ\tilde{h}$ and the radial limit function of f agree almost everywhere, and so

$$f(z) - \alpha z - \overline{\beta z} - A\log|z| = -\frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\left[\frac{e^{i\theta}+z}{e^{i\theta}-z}\right](\phi\circ\tilde{h}(e^{i\theta}) - \alpha e^{i\theta} - \overline{\beta}e^{-i\theta})\,d\theta.$$

From Lemma 2.1(a), we know that the unrestricted limits

$$\operatorname{Lim}_{z\to e^{i\theta}}[f(z) - \alpha z - \overline{\beta z} - A\log|z|]$$

exist and are equal to $\phi \circ \tilde{h} - \alpha e^{i\theta} - \overline{\beta} e^{-i\theta}$ at all points of $(\partial\Delta)\backslash E_0$. Therefore we conclude that the unrestricted limits $\hat{f}(e^{i\theta}) \equiv \mathrm{Lim}_{z \to e^{i\theta}} f(z)$ exist and are equal to $\phi \circ \tilde{h}$ at all points of $(\partial\Delta)\backslash E_0$.

Next, let $e^{i\theta_0}$ belong to E_0. If z approaches $e^{i\theta_0}$ along $L_\psi(e^{i\theta_0})$ or any path tangent to $L_\psi(e^{i\theta_0})$ at $e^{i\theta_0}$ $(0 < \psi < \pi)$, then $f(z)$ converges to the value $\psi A_0/\pi + (\pi - \psi)B_0/\pi$, where $A_0 = \mathrm{Lim}_{\theta \uparrow \theta_0} \hat{f}(e^{i\theta})$ and $B_0 = \mathrm{Lim}_{\theta \downarrow \theta_0} \hat{f}(e^{i\theta})$. Therefore the cluster set of f at θ_0 is the straight-line segment joining A_0 to B_0. If the one-sided limits A_0 and B_0 are equal, then the cluster set is a singleton; so f has a limit and $\hat{f}$ is continuous there. If they are different (i.e., $A_0 \neq B_0$), then Γ would have to contain line segments corresponding to points of the discontinuity set $(\subset E_0)$. Since U^c is assumed to be strictly convex, the discontinuity set must be empty. Therefore f extends continuously to $\overline{\Delta}$.

Remark. By Theorem 2.2, we know that if $f \in \Sigma_H$ and $f(\Delta) = \Delta$, then f has a continuous extension to $\overline{\Delta}$ and f has a Poisson integral representation

$$f(z) = \alpha z + \overline{\beta z} + A \log|z| - \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Re}\left[\frac{e^{it} + z}{e^{it} - z}\right] (e^{i\theta(t)} - \alpha e^{it} - \overline{\beta} e^{-it})\, dt$$

where θ is a nondecreasing continuous function on $\mathbb{R}$ with $\theta(t + 2\pi) = \theta(t) + 2\pi$.

Recall that if $f \in \Sigma_H$ then f admits the representation (1.1). We now consider such functions and obtain sharp estimates for Fourier coefficients (1.1). Before doing this, let us consider harmonic functions defined on the open unit disk for the sake of comparison.

The following Theorem 2.3 is proved by Duren and Schober [2]; but it can be more easily proved by using the integration by parts. So we will give the proof here.

Theorem 2.3. *Let f be an orientation-preserving univalent harmonic mapping of the unit disk onto itself with the Fourier expansion*

$$f(z) = \sum_{n=1}^{\infty} C_{-n} \overline{z}^n + \sum_{n=0}^{\infty} C_n z^n .$$

Then $|C_n| \leq 1/n$, $n = 1, 2, \ldots$.

Proof. Since f has a continuous extension to $\{z : |z| \leq 1\}$, which defines a weak homeomorphism of $|z| = 1$ onto itself (cf. [1]), for each $n \geq 1$ we have

$$\begin{aligned} \left| \int_{|z|=1} z^{-n}\, df \right| &= \left| \int_0^{2\pi} e^{-in\theta}\, df(e^{i\theta}) \right| \\ &= \left| e^{-in\theta} f(e^{i\theta})|_0^{2\pi} + in \int_0^{2\pi} f(e^{i\theta}) e^{-in\theta}\, d\theta \right| = 2n\pi |C_n| \end{aligned}$$

by integration by parts. Since $|\int_{|z|=1} z^{-n}\, df| \leq \int_{|z|=1} |\, df| = 2\pi$, we have $|C_n| \leq 1/n$, $n \geq 1$.

It is shown in [2] that the estimate of Theorem 2.3 is sharp for each n. By analyzing the case of equality, we could obtain the extremal functions, too.

The following applies to the class Σ_H.

Theorem 2.4. *If $f \in \Sigma_H$ and f extends to be of bounded variation on $|z| = 1$, then*

$$|\alpha + \overline{b}_1| \le L/2\pi, \qquad |b_n| \le L/2n\pi \quad \text{for } n \ge 2,$$
$$|\beta + \overline{a}_1| \le L/2\pi, \qquad |a_n| \le L/2n\pi \quad \text{for } n \ge 2,$$

where L is the length of $f(|z| = 1)$.

The first inequality $|\alpha + \overline{b}_1| \le L/2\pi$ is sharp for the function

$$f(z) = z + b_1/\overline{z} + A \log|z| \tag{2.1}$$

whenever $|b_1| < 1$ and $|A| \le (1 - |b_1|^2)/|1 + b_1|$, $|b_1| = 1$ and $A = 0$, or $b_1 = -1$ and $|A| \le 2$.

The inequality $|\beta + \overline{a}_1| \le L/2\pi$ is sharp for the function

$$f(z) = z - 1/\overline{z} + 2\log|z|. \tag{2.2}$$

The inequalities $|b_n| \le L/2n\pi$ and $|a_n| \le L/2n\pi$ (for $n \ge 2$) are sharp for the function

$$f(z) = z - 1/\overline{z} + 2\arg((1 + i/z)/(1 - i/z)). \tag{2.3}$$

Proof. By integration by parts, we have

$$\left|\int_{|z|=1} z^n\, df\right| = \begin{cases} 2\pi|\alpha + \overline{b}_1| & \text{if } n = -1, \\ 2\pi|-n\overline{b}_{-n}| & \text{if } n \le -2, \\ 2\pi|a_1 + \overline{\beta}| & \text{if } n = 1, \\ 2\pi n|a_n| & \text{if } n \ge 2. \end{cases}$$

Since $|\int_{|z|=1} z^n\, df| \le \int_{|z|=1} |df| = L$, we have

$$|\alpha + \overline{b}_1| \le L/2\pi, \qquad |a_1 + \overline{\beta}| \le L/2\pi,$$

and

$$n|b_n| \le L/2\pi, \quad n|a_n| \le L/2\pi \qquad \text{for } n \ge 2.$$

These inequalities are equivalent to the desired ones.

For the given parameters b_1 and A the function (2.1) maps Δ onto the exterior of the circle $|w| = |1 + b_1|$. The function (2.2) maps Δ onto $\mathbb{C}\backslash\{0\}$. The function (2.3) maps Δ onto the complement of real line segment $[-\pi, \pi]$. For these functions, the bounds are sharp. To see that the mappings are univalent note that they are local homeomorphisms since their Jacobians are positive and they are one-to-one on circles near $\partial\Delta$. Also, to see that the mappings (2.1) and (2.2) are univalent, observe that they map the circles $|z| = R$, $R > 1$, onto an increasing family of (possibly nonconcentric) circles.

Remark. If $E = \mathbb{C}\backslash f(\Delta)$ is strictly convex, then f has a continuous extension to $\overline{\Delta}$ by Theorem 2.2. In this case the bounded variation hypothesis is unnecessary in Theorem 2.4 and L is the length of ∂E.

Corollary 2.5. *If $f \in \Sigma_H$ and $f(\Delta) = \Delta$, then*

$$\begin{aligned} &|\alpha + \overline{b}_1| \le 1, \qquad |b_n| \le 1/n \quad \text{for } n \ge 2, \\ &|\beta + \overline{a}_1| \le 1, \qquad |a_n| \le 1/n \quad \text{for } n \ge 2. \end{aligned} \tag{2.4}$$

The first inequality is sharp for the functions

$$f(z) = z + e^{i\gamma}/\overline{z} \quad \textit{and} \quad f(z) = z + A\log|z|,$$

where $\gamma = \pm 2\pi/3, \pm 4\pi/3$ *and* $|A| \le 1$.

Proof. Since the unit circle has length $L = 2\pi$, the inequalities follow directly from Theorem 2.4 and the remark above. The extremal mappings for the first inequality are special cases of the mappings (2.1) in Theorem 2.4.

In §3, we will show that the bounds for the coefficients b_n in (2.4) of Corollary 2.5 are sharp.

3. Automorphisms of $\Delta = \{z : |z| > 1\}$

For each $n \ge 2$, consider the function $e^{i\theta_n(t)}$ where

$$\theta_n(t) = \begin{cases} nt & \text{if } 0 \le t \le 2\pi/n, \\ 2\pi & \text{if } 2\pi/n \le t \le 2\pi. \end{cases}$$

Then

$$(3.1)\quad f(z) = \alpha z + \overline{\beta z} + A\log|z| - \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\left[\frac{e^{it}+z}{e^{it}-z}\right](e^{i\theta_n(t)} - \alpha e^{it} - \overline{\beta e^{it}})\,dt$$

is a harmonic function in Δ and $f(\infty) = \infty$. In addition, $f(z)$ has a series expansion (1.1) and $b_n = 1/n$.

Let $\alpha = ke^{i(n-1)\pi/n}$, $A = k$, and

$$\beta = \begin{cases} 0 & \text{for } n = 2, \\ k(\frac{2}{3} - \frac{1}{6}\sec\frac{\pi}{n})e^{-i\pi/n} & \text{for } n \ge 3. \end{cases}$$

Then the necessary conditions $0 \le |\beta| < |\alpha|$ and $|A|/2 \le |\alpha| + |\beta|$ for the mapping in Σ_H are satisfied for all $k > 0$. We made these choices for α, β, and A with the help of a computer. In the remainder of this section we shall show that there exists a $k > 0$ so that f is univalent, hence an extremal function in Σ_H.

Basic properties of complex variables and lengthy, but straightforward, calculations give the following lemma.

Lemma 3.1. *For all* k *sufficiently large, the function* (3.1) *satisfies*

(a) $|f_z| = |f_{\overline{z}}|$ *on* $\Gamma_0 = \{z = e^{i\theta} : 2\pi/n < \theta < 2\pi\}$,
$|f_z| > |f_{\overline{z}}|$ *on* $\Gamma = \{z = e^{i\theta} : 0 < \theta < 2\pi/n\}$ *and*

(b) $f_z \ne 0$ *on* $D_0 = \{z : |z| \ge 1,\ z \ne 1,\ z \ne e^{i2\pi/n}\}$.

Proof of this lemma is elementary, so it may be left to the reader.

Lemma 3.2. $f(z)$ *is locally univalent in* Δ *for all* k *sufficiently large.*

Proof. From Lemma 3.1, we know that $|f_z| \ge |f_{\overline{z}}|$ on $\Gamma_0 \cup \Gamma$ and $f_z \ne 0$ on D_0 for all k sufficiently large. Let $\tilde{a}(z) = \overline{f_{\overline{z}}}/f_z$. Then $\tilde{a}(z)$ is analytic on Δ and $|\tilde{a}(z)| \le 1$ on $\Gamma_0 \cup \Gamma$.

$$|\tilde{a}(z)| = \left|\frac{\overline{f_{\overline{z}}}}{f_z}\right| = \left|\frac{(\text{Regular terms})/L(z) - in(2\pi)^{-1}z^{-n-1}}{(\text{Regular terms})/L(z) + in(2\pi)^{-1}z^{n-1}}\right| \to 1$$

as $z \to 1$ or $z \to e^{i2\pi/n}$

where $L(z) = \log[(z - e^{i2\pi/n})/(z-1)]$. By the Maximum Principle, we have $|\tilde{a}(z)| \le 1$ in Δ. If $|\tilde{a}| = 1$ at some point in Δ, then $|\tilde{a}| \equiv 1$; but $|\tilde{a}(\infty)| = |\beta/\alpha| < \frac{1}{2}$. This implies that $|\tilde{a}| < 1$ in Δ. Thus $|f_z| > |\overline{f_{\bar{z}}}|$ in Δ. Therefore f is locally univalent in Δ, and at ∞, too.

Lemma 3.3. *Let S^2 be the Riemann sphere. If $F: S^2 \to S^2$ is a local homeomorphism of $S^2 \backslash \{\infty\}$ and continuous on S^2, then F is a global homeomorphism of S^2 onto S^2.*

Proof. Without loss of generality, we may assume that $F(\infty) = \infty$. Since F is continuous and S^2 is compact, $F: S^2 \to S^2$ is a proper map (i.e., the inverse image of any compact set is compact). Put $\mathbb{C}' = F^{-1}(\mathbb{C}) = \mathbb{C} \backslash F^{-1}(\infty)$. Then $F: \mathbb{C}' \to \mathbb{C}$ is both a local homeomorphism and a proper map, so it is a covering map. Now $\mathbb{C}$ is simply connected and $\mathbb{C}'$ is the complement in $\mathbb{C}$ of the discrete subset $\mathbb{C} \cap F^{-1}(\infty)$ of $\mathbb{C}$. Therefore the covering map $F: \mathbb{C}' \to \mathbb{C}$ is a homeomorphism, $\mathbb{C}' = \mathbb{C}$, and $F: \mathbb{C} \to \mathbb{C}$ is a homeomorphism; hence, so is $F: S^2 \to S^2$.

Theorem 3.4. *For each $n \ge 2$, there exists a harmonic, orientation-preserving, univalent mapping f of Δ onto itself with the Fourier expansion* (1.1) *such that $b_n = 1/n$.*

Proof. For each n, take (3.1) with $\alpha = ke^{i(n-1)\pi/n}$, $\beta = k\xi_n e^{-i\pi/n}$, and $A = k$ where ξ_n is $\frac{2}{3} - \frac{1}{6}\sec(\pi/n)$ if $n \ge 3$ and 0 if $n = 2$. Then there exists $k > 0$ such that f is a local homeomorphism on $\Delta \cup \{\infty\}$ and the Jacobian of f, $\mathrm{J}_f = |f_z|^2 - |f_{\bar{z}}|^2$, is positive on $\Gamma = \{z = e^{i\theta}: 0 < \theta < 2\pi/n\}$ by Lemma 3.2 and Lemma 3.1. f is a local homeomorphism in the full neighborhood of each point p of Γ to some neighborhood of $f(p)$ since $\mathrm{J}_f > 0$ on Γ.

Now define the reflection

$$G(z) = \begin{cases} f(z) & \text{if } |z| \ge 1, \\ 1/\overline{f(1/\bar{z})} & \text{if } |z| < 1. \end{cases}$$

Then G is a local homeomorphism at each point of $S^2 \backslash \tilde{\Gamma}$ where $\tilde{\Gamma} = \{z = e^{i\theta}: 2\pi/n \le \theta \le 2\pi\}$. It is continuous on S^2; however, G is constant on $\tilde{\Gamma}$. Now identify points of $\tilde{\Gamma}$ and call this element b. We obtain a new function F on a new domain, which is again topologically a sphere S^2, and F is a local homeomorphism on $S^2 \backslash \{b\}$ and continuous on S^2. Apply Lemma 3.3. Then F is a homeomorphism. Hence G is a homeomorphism on $S^2 \backslash \tilde{\Gamma}$. Therefore $G|_\Delta = f$ is a homeomorphism.

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