PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 119, Number 1, September 1993

THE HANKEL TRANSFORMATION OF BANACH-SPACE-VALUED GENERALIZED FUNCTIONS

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. The object of this paper is to study Banach-space-valued generalized functions belonging to $[H_\mu(A); B]$ for which the Hankel transformation may be defined. In *Realizability theory for continuous linear systems* (Academic Press, New York, 1972), Zemanian considered certain ρ-type testing function spaces for which the Laplace transformation is defined. Tiwari (*Banach space valued distributional Mellin transform and form invariant linear filtering*, Indian J. Pure Appl. Math. **20** (1989), 493–504) follows Zemanian in extending the Mellin transform. Their works are based on the denseness of the Schwartz space $D^m(A)$ in the testing function spaces of interest. This method is not possible here since the space $D^m(A)$ is not dense in $H_\mu(A)$, and the structure of $H_\mu(A)$ is quite different from that of $D^m(A)$, which has an inductive-limit topology. Thus, it is necessary to introduce a dense subspace ${}_\mu D_I(A)$ of $H_\mu(A)$ to derive some properties of $H_\mu(A)$. We then define the Hankel transformation on $[H_\mu(A); B]$. We end this paper with some operational formulas, which are analogous with those given by the first author in SIAM J. Math. Anal. **1** (1970), 322–327.

1. INTRODUCTION

Zemanian studied the theory of Banach-space-valued testing functions and distributions in [2], which is somewhat more general than that of scalar distributions. He constructed $D^m(A)$ as the inductive-limit space given by

$$D^m(A) = D^m_{R^n}(A) = \bigcup_{j=1}^{\infty} D^m_{K_j}(A)$$

where $D^m_{K_j}(A)$ denotes the linear space of all smooth functions ϕ from R^n into a Banach space A such that $\operatorname{supp}\varphi \subset K_j$. K_j are compact subsets of R^n and $K_j \subset K_{j+1}$, $\bigcup_{j=1}^\infty K_j = R^n$. We assign to $D^m_{K_j}(A)$ the topology generated by the collection $\{\gamma_k;\ 0 \le k \le m\}$ of seminorms, where $\gamma_k(\phi) \triangleq \sup_{t\in K_j} \|\phi^{(k)}(t)\|_A$.

Applying the interpolation theory, he describes the local structure property below.

Received by the editors January 21, 1992; this paper was presented on December 10, 1991, at the winter meeting of the Canadian Mathematical Society in Victoria, Canada.

1991 *Mathematics Subject Classification*. Primary 46F10.

Key words and phrases. The Hankel transformation, inductive-limit topology, generalized functions, Banach space.

Theorem 1.1. *Let $f \in [D^m(A); B]$ and K be a compact interval in R^n. Then there exists an integer $p \in R^n$ with $0 \leq p \leq m$ and a continuous $[A; B]$-valued function h on K such that, for all $\phi \in D_K^{m+[2]}(A)$,*

$$\langle f, \phi \rangle = \int_K h(t) D^{p+[2]} \phi(t)\, dt .$$

In general, p and h depend on f and K.

Finally, he mentions

Theorem 1.2. *If $\mathscr{T}^m$ and $\mathscr{T}^m(A)$ are normal spaces (i.e., D is dense in $\mathscr{T}^m$ and $\mathscr{T}^m(A)$), then there exists a bijection from $[\mathscr{T}^m(A); B]$ onto $[\mathscr{T}^m; [A; B]]$ defined by*

$$\langle g, \psi \rangle a \triangleq \langle f, \psi a \rangle, \qquad \psi \in \mathscr{T}^m, \; a \in A,$$

where $g \in [\mathscr{T}^m; [A; B]]$, $f \in [\mathscr{T}^m(A); B]$.

Tiwari [3] mimics the method of Zemanian in defining Banach-space-valued distributions for which a Mellin transform can be given. Several properties including a Mellin-type convolution theorem are proved. These results are similar to those of Zemanian [1].

In this paper, we introduce a dense subspace ${}_\mu D_I(A)$ of $H_\mu(A)$. It does not have an inductive-limit topology. The local structure theorem is no longer discussed in $[H_\mu(A); B]$. However, with a different method, we show that there is still a bijection from $[H_\mu(A); B]$ onto $[H_\mu; [A; B]]$. Further, we are able to define the Hankel transformation of arbitrary order on $H_\mu(A)$, which is still an automorphism on $H_\mu(A)$. We give some operational formulas at the end of this paper.

Our notation is similar to that used in [1, 2, 4]. Given any two topological vector spaces A and B, $[A; B]$ denotes the linear space of all continuous linear mappings of A into B. The element of B assigned by $f \in [A; B]$ to $\phi \in A$ is denoted by $\langle f, \phi \rangle$. $[A; B]$ is supplied with the topology of uniform convergence on bounded sets in A. $\| \cdot \|_B$ denotes the norm in any Banach space B. R and C are the real and complex number fields. I is the open interval $(0, +\infty)$. Other notation will be introduced as the need arises.

2. The correspondence between $[H_\mu(A); B]$ and $[H_\mu; [A; B]]$ for $\mu \geq -\frac{1}{2}$

Following Zemanian, $H_\mu(A)$ is defined as follows:

Definition 2.1. Let x be a real variable restricted to I. For each real number μ, $\phi(x) \in H_\mu(A)$ iff it is defined on I, takes its values in A, is smooth, and for each pair of nonnegative integers m and k

$$\gamma_{m,k}^\mu(\phi) \triangleq \sup_{x \in I} \| x^m (x^{-1} D)^k x^{-\mu - 1/2} \phi(x) \|_A$$

is finite. $H_\mu(A)$ is a linear space. The topology of $H_\mu(A)$ is that generated by $\{\gamma_{m,k}^\mu\}_{m,k=0}^\infty$.

Definition 2.2. $\phi(x) \in D_I(A)$ iff ϕ is defined on I, takes its value in A, is smooth, and for every ϕ there exists $b \in I$ such that $\phi(x) = 0$ for $x \in [b, +\infty)$. Let ${}_\mu D_I(A) \triangleq D_I(A) \cap H_\mu(A)$.

Theorem 2.1. *The subspace* ${}_\mu D_I(A)$ *is dense in* $H_\mu(A)$ *for all* $\mu \in R$.

Proof. Let $\lambda(x) \in D_I(C)$ such that $\lambda(x) = 1$ for $0 < x \leq 1$ and $\lambda(x) = 0$ for $x \geq 2$.

For arbitrary $\phi(x) \in H_\mu(A)$ and each pair of nonnegative integers m and k we consider

$$x^m(x^{-1}D)^k x^{-\mu-1/2}[\lambda(x/N)\phi(x) - \phi(x)]$$
$$= x^{m+1}\sum_{v=0}^{k}\binom{k}{v}(x^{-1}D)^{k-v}x^{-\mu-1/2}\phi\frac{(x^{-1}D)^v[\lambda(x/N)-1]}{x}.$$

Therefore,

$$\sup_{x\in I}\|x^m(x^{-1}D)^k x^{-\mu-1/2}(\lambda(x/N)\phi(x)-\phi(x))\|_A$$
$$\leq \sum_{v=0}^{k}\binom{k}{v}\sup_{x\in I}\|x^{m+1}(x^{-1}D)^{k-v}x^{-\mu-1/2}\phi\|_A \cdot \sup_{x\geq N}\left|\frac{(x^{-1}D)^v[\lambda(x/N)-1]}{x}\right|.$$

It follows from $\phi \in H_\mu(A)$ that

$$\sup_{x\in I}\|(x^{-1}D)^{k-v}x^{-\mu-1/2}\phi\|_A$$

is finite.

Since $\lambda(x)$ and its derivatives are bounded, it follows that

$$\sup_{x\geq N}\left|\frac{(x^{-1}D)^v[\lambda(x/N)-1]}{x}\right| \to 0 \quad \text{as } N\to\infty, \quad \text{for fixed } k \text{ and } 0\leq v\leq k,$$

whence our assertion.

$H_\mu(A)$ is not a ρ-type testing function space in the sense of Zemanian [2]. To see this, we choose $\phi(x) = x^{\mu+1/2}e^{-x^2}a_0$, $a_0 \in A$ and $a_0 \neq 0$. Then for all ψ, which is smooth from I into A with compact support contained in I, $\gamma^\mu_{0,0}(\phi-\psi) \geq \|a_0\|/2 > 0$. This means the balloon

$$\{\theta;\ \theta\in H_\mu(A),\ \gamma^\mu_{0,0}(\phi-\theta)\leq \|a_0\|/3\}$$

does not contain any element of $D^m(A)$. Thus our result is true.

The following lemmas will be used subsequently (see [1, 2]).

Lemma 2.1. *Let* V, W *be locally convex spaces, and let* Γ *and* P *generate families of seminorms of the topologies of* V *and* W, *respectively. Let* f *be a linear mapping of* V *into* W. *The following four assertions are equivalent*:

(i) f *is continuous.*

(ii) f *is continuous at the origin.*

(iii) *For every continuous seminorm* ρ *on* W, *there exists a continuous seminorm* γ *on* V *such that* $\rho(f(\theta)) \leq \gamma(\theta)$ *for all* θ.

(iv) *For every* $\rho \in P$, *there exists a constant* $M > 0$ *and a finite collection* $\{\gamma_1, \gamma_2, \dots, \gamma_m\} \subset \Gamma$ *such that*

$$\rho(f(\theta)) \leq M \max_{0\leq k\leq m}\gamma_k(\theta)$$

for all $\theta \in V$.

Lemma 2.2. *For* $\mu \geq -\frac{1}{2}$, *the conventional* h_μ *is an automorphism on* $H_\mu(A)$.

Proof. Very similar to Theorem 5.4.–1, p. 141 of [1].

Theorem 2.2. *Every* $f \in [H_\mu(A); B]$ *uniquely defines a* $g \in [H_\mu; [A; B]]$ *through the equation*

$$\langle g, \theta \rangle a \triangleq \langle f, \theta a \rangle, \qquad \theta \in H_\mu, \ a \in A,$$

for all $\mu \in R$.

Proof. Fixing upon some $\theta \in H_\mu$ we define a mapping j_θ of A into B by $j_\theta a = \langle f, \theta a \rangle$ for all $a \in A$. It readily follows that j_θ is linear. By Lemma 2.1(iv) there exist positive integers m_0, k_0 and a constant $M > 0$ such that

$$\|j_\theta a\|_B = \|\langle f, \theta a\rangle\|_B \le M \max_{\substack{0\le k\le k_0 \\ 0\le m\le m_0}} \gamma^\mu_{m,k}(\theta a)$$

where

$$\gamma^\mu_{m,k}(\theta a) = \sup_{x\in I} \|x^m (x^{-1}D)^k x^{-\mu-1/2}\theta a\|_A$$
$$= \|a\|_A \sup_{x\in I} |x^m (x^{-1}D)^k x^{-\mu-1/2}\theta|.$$

Hence

$$\|j_\theta a\|_B \le M\|a\|_A \max_{\substack{0\le k\le k_0 \\ 0\le m\le m_0}} \gamma^\mu_{m,k}(\theta)$$

and

$$\|j_\theta\|_{[A;B]} \le M \max_{\substack{0\le k\le k_0 \\ 0\le m\le m_0}} \gamma^\mu_{m,k}(\theta). \tag{2.1}$$

Next, set $\langle g, \theta\rangle \triangleq j_\theta$. This uniquely defines g as a mapping from H_μ into $[A; B]$. g is linear because, for any $a \in A$, $\alpha, \beta \in C$, and $\theta, \psi \in H_\mu$

$$\langle g, \alpha\theta + \beta\psi\rangle a = \langle f, \alpha\theta a + \beta\psi a\rangle = \alpha\langle f, \theta a\rangle + \beta\langle f, \psi a\rangle$$
$$= (\alpha\langle g, \theta\rangle + \beta\langle g, \psi\rangle)a.$$

Moreover, (2.1) implies that g is continuous.

Let ${}_\mu D_I \odot A$ denote the linear space of all $\varphi \in {}_\mu D_I(A)$ having representations of the form $\varphi = \sum \theta_k a_k$ where $\theta_k \in {}_\mu D_I$, $a_k \in A$, and the summation is over a finite number of terms.

Theorem 2.3. *The space* ${}_\mu D_I \odot A$ *is dense in* $H_\mu(A)$ *for* $\mu \ge -\frac{1}{2}$.

Proof. Let $\lambda(x)$ be defined as in the proof of Theorem 2.1. For $\varphi \in {}_\mu D_I(A)$, we first show that

$$\lambda(x/N)h_\mu(\varphi) \to h_\mu(\varphi) \quad \text{in } H_\mu(A) \text{ as } N \to +\infty \text{ for all } \mu \in R.$$

The following equation will be used (see [1]):

$$(-1)^{m+k} y^m (y^{-1}D)^k y^{-\mu-1/2} h_\mu(\varphi)(y)$$
$$= \int_0^{+\infty} x^{2\mu+2k+m+1}[(x^{-1}D)^m x^{-\mu-1/2}\varphi(x)] \frac{J_{\mu+k+m}(xy)}{(xy)^{\mu+k}}\, dx. \tag{2.2}$$

Hence

$$\sup_{x\in I} \left\| x^m (x^{-1}D)^k x^{-\mu-1/2} h_\mu(\varphi)\left[\lambda\left(\frac{x}{N}\right) - 1\right]\right\|_A$$
$$\le \sum_{v=0}^{k} \binom{k}{v} \sup_{x\ge N}\left|\frac{(x^{-1}D)^v[\lambda(x/N) - 1]}{x}\right| \sup_{x\in I} \|x^{m+1}(x^{-1}D)^{k-v} x^{-\mu-1/2} h_\mu(\varphi)\|_A.$$

By what we have proved in Theorem 2.1,

$$\sup_{x\geq N}\left|\frac{(x^{-1}D)^v[\lambda(x/N)-1]}{x}\right| \to 0 \quad \text{as } N\to\infty \text{ for fixed } k \text{ and } 0\leq v\leq k .$$

Using equation (2.2) and noting that $J_{\mu+k-v+m+1}(xy)/(xy)^{\mu+k-v}$ is bounded, say by $B_{k,v,m}$, we get

$$\begin{aligned}\sup_{x\in I}&\|x^{m+1}(x^{-1}D)^{k-v}x^{-\mu-1/2}h_\mu(\varphi)\|_A\\ &=\sup_{x\in I}\left\|\int_0^{+\infty}y^{2\mu+2(k-v)+m+2}[(y^{-1}D)^{m+1}y^{-\mu-1/2}\varphi(y)]\frac{J_{\mu+k-v+m+1}(xy)}{(xy)^{\mu+k-v}}\,dy\right\|_A .\end{aligned}$$

Choosing a positive integer n such that

$$y^{2\mu+2(k-v)+m+2}\leq(1+y^2)^n \quad \text{for all } y\in I$$

we have

$$\begin{aligned}\sup_{y\in I}&\|y^{2\mu+2(k-v)+m+2}[(y^{-1}D)^{m+1}y^{-\mu-1/2}\varphi(y)]\|_A\\ &\leq\sup_{y\in I}\|(1+y^2)^n[(y^{-1}D)^{m+1}y^{-\mu-1/2}\varphi(y)]\|_A .\end{aligned}$$

Since $\varphi\in {}_\mu D_I(A)$, there exists $b\in I$ such that $\varphi(x)=0$ for $x\in[b,+\infty)$. It follows that

$$\begin{aligned}\sup_{x\in I}&\|x^{m+1}(x^{-1}D)^{k-v}x^{-\mu-1/2}h_\mu(\varphi)\|_A\\ &\leq B_{k,v,m}b\sup_{y\in I}\|(1+y^2)^n[(y^{-1}D)^{m+1}y^{-\mu-1/2}\varphi(y)]\|_A\end{aligned}$$

is finite. Therefore,

$$\lambda(x/N)h_\mu(\varphi)\to h_\mu(\varphi) \quad \text{in } H_\mu(A) \text{ as } N\to+\infty .$$

Second, we prove that ${}_\mu D_I\odot A$ is dense in $H_\mu(A)$ for $\mu\geq-\frac{1}{2}$. For a positive integer m_1, we have

$$\sqrt{xy}J_\mu(xy)=\sum_{j=0}^{m_1}\frac{(xy)^{1/2}(-1)^j(xy/2)^{\mu+2j}}{j!\,\Gamma(\mu+j+1)}+\sum_{j=m_1+1}^{\infty}\frac{(xy)^{1/2}(-1)^j(xy/2)^{\mu+2j}}{j!\,\Gamma(\mu+j+1)} .$$

For every $\varphi\in {}_\mu D_I(A)$, the term

$$T_{N,m_1}=\lambda\left(\frac{x}{N}\right)\int_0^{+\infty}\varphi(t)\sum_{j=0}^{m_1}\frac{(xt)^{1/2}(-1)^j(xt/2)^{\mu+2j}}{j!\,\Gamma(\mu+j+1)}\,dt,$$

where $N, m_1=1,2,\ldots$, belongs to ${}_\mu D_I\odot A$ since $\mu\geq-\frac{1}{2}$. Now

$$\begin{aligned}T_{N,m_1}&-\int_0^{+\infty}\varphi(t)\sqrt{xt}J_\mu(xt)\,dt\\ &=T_{N,m_1}-\lambda\left(\frac{x}{N}\right)\int_0^{+\infty}\varphi(t)\sqrt{xt}J_\mu(xt)\,dt\\ &\quad+\lambda\left(\frac{x}{N}\right)\int_0^{+\infty}\varphi(t)\sqrt{xt}J_\mu(xt)\,dt-\int_0^{+\infty}\varphi(t)\sqrt{xt}J_\mu(xt)\,dt .\end{aligned}$$

By what we have just proved, for arbitrary $\varepsilon > 0$, there exists an N_1 such that for $N \geq N_1$, we have

$$\sup_{x\in I} \|x^m(x^{-1}D)^k x^{-\mu-1/2}[\lambda(x/N)h_\mu(\varphi) - h_\mu(\varphi)]\|_A < \varepsilon/2.$$

Fixing N $(\geq N_1)$, then

$$\lambda\left(\frac{x}{N}\right)\left[\sum_{j=0}^{m_1} \frac{(xt)^{1/2}(-1)^j(xt/2)^{\mu+2j}}{j!\,\Gamma(\mu+j+1)} - \sqrt{xt}J_\mu(xt)\right]$$

and its derivatives with respect to x converge to zero uniformly on every compact subset of I. It has a uniformly bounded support. Therefore it converges in the sense of Schwartz, whose topology is stronger than that of H_μ (see [1]). It follows that there exists an $L \in I$ such that as long as $m_1 \geq L$, then for all $t \leq b$,

$$\sup_{x\in I}\left|x^m(x^{-1}D)^k x^{-\mu-1/2}\lambda\left(\frac{x}{N}\right)\left[\sum_{j=0}^{m_1} \frac{(xt)^{1/2}(-1)^j(xt/2)^{\mu+2j}}{j!\,\Gamma(\mu+j+1)} - \sqrt{xt}J_\mu(xt)\right]\right| \leq \frac{\varepsilon}{2M_1},$$

where $M_1 = b\sup_{t\in I}\|\varphi(t)\|_A$. If $M_1 = 0$, then there is nothing to be proved. Therefore,

$$\sup_{x\in I}\left\|x^m(x^{-1}D)^k x^{-\mu-1/2}\left[T_{N,m_1} - \int_0^{+\infty}\varphi(t)\sqrt{xt}J_\mu(xt)\,dt\right]\right\|_A < \varepsilon$$

provided $N \geq N_1$, $m_1 \geq L$.

Since h_μ is an automorphism on $H_\mu(A)$ for $\mu \geq -\frac{1}{2}$ by Lemma 2.2 and ${}_\mu D_I(A)$ is dense in $H_\mu(A)$, it follows that $h_\mu({}_\mu D_I(A))$ is dense in $H_\mu(A)$. Our assertion follows directly from the fact that ${}_\mu D_I \odot A$ is dense in $h_\mu({}_\mu D_I(A))$.

Theorem 2.4. *There is a bijection from* $[H_\mu(A); B]$ *onto* $[H_\mu; [A; B]]$ *defined by*

$$\langle g, \theta\rangle a = \langle f, \theta a\rangle$$

where $a \in A$, $g \in [H_\mu; [A; B]]$, *and* $f \in [H_\mu(A); B]$, $\theta \in H_\mu$ *for* $\mu \geq -\frac{1}{2}$.

Proof. By Theorem 2.2, every $f \in [H_\mu(A); B]$ uniquely defines a $g \in [H_\mu; [A; B]]$ through the equation

$$\langle g, \theta\rangle a \triangleq \langle f, \theta a\rangle \quad \text{for all } \mu \in R.$$

Let us consider the converse. For every $\varphi \in {}_\mu D_I \odot A$, we define

$$\langle f, \varphi\rangle = \sum \langle g, \theta_k\rangle a_k \quad \text{for } \varphi = \sum \theta_k a_k.$$

It follows from the definition that f is linear on ${}_\mu D_I \odot A$. We wish to show that f is continuous. Indeed, for arbitrary $\varepsilon > 0$, as long as θa $(\theta \in {}_\mu D_I$, $a \in A)$ belongs to the balloon $\{\varphi;\ \gamma^\mu_{m,k}(\varphi) < \varepsilon/M,\ m = 0, 1, \dots, m_0,\ k = 0, 1, \dots, k_0\}$. M, m_0, k_0 are defined as follows. We infer that

$$\|\langle f, \theta a\rangle\|_B = \|\langle g, \theta\rangle a\|_B \leq \|a\|_A \cdot \|\langle g, \theta\rangle\|_{[A;B]}.$$

By Lemma 2.1(iv), there exist $M > 0$ and positive integers m_0, k_0 such that

$$\|\langle f, \theta a\rangle\|_B \leq \|a\|_A \cdot M \max_{\substack{0\leq k\leq k_0 \\ 0\leq m\leq m_0}} \gamma^{\mu}_{m,k}(\theta) < M \cdot \varepsilon/M = \varepsilon .$$

Therefore f is continuous at the origin. By Lemma 2.1(ii), f is continuous on ${}_{\mu}D_I \odot A$. According to Theorem 2.3, ${}_{\mu}D_I \odot A$ is dense in $H_{\mu}(A)$ for $\mu \geq -\frac{1}{2}$. Thus our assertion is true.

3. The Hankel transformation h_{μ} on $H_{\mu}(A)$ and the correspondence between $[H_{\mu}(A); B]$ and $[H_{\mu}; [A; B]]$ for arbitrary μ

We shall use the following differential and integral operators due to Zemanian [1]:

$$N_{\mu}\varphi(x) \triangleq x^{\mu+1/2} D x^{-\mu-1/2}\varphi(x),$$
$$M_{\mu}\varphi(x) \triangleq x^{-\mu-1/2} D x^{\mu+1/2}\varphi(x),$$
$$N_{\mu}^{-1}\varphi(x) \triangleq x^{\mu+1/2}\int_{\infty}^{x} t^{-\mu-1/2}\varphi(t)\,dt .$$

Lemma 3.1. *N_{μ} is a continuous linear mapping of $H_{\mu}(A)$ into $H_{\mu+1}(A)$.*

Indeed, $\gamma^{\mu+1}_{m,k}(N_{\mu}\varphi) = \gamma^{u}_{m,k+1}(\varphi)$ for every $\varphi \in H_{\mu}(A)$ and every choice of m and k.

Lemma 3.2. *N_{μ}^{-1} is a continuous linear mapping of $H_{\mu+1}(A)$ into $H_{\mu}(A)$.*

Proof. Assume that $\varphi(x) \in H_{\mu+1}(A)$ and k is a fixed positive integer. Then

$$\begin{aligned}(x^{-1}D)^k x^{-\mu-1/2} N_{\mu}^{-1}\varphi(x) &= (x^{-1}D)^k x^{-\mu-1/2} x^{\mu+1/2}\int_{\infty}^{x} t^{-\mu-1/2}\varphi(t)\,dt \\ &= (x^{-1}D)^{k-1} x^{-\mu-3/2}\varphi(x).\end{aligned}$$

Hence

$$\gamma^{\mu}_{m,k}(N_{\mu}^{-1}\varphi) = \gamma^{\mu+1}_{m,k-1}(\varphi), \qquad k = 1, 2, 3, \ldots, \quad m = 0, 1, 2, \ldots .$$

A similar result for the case $k = 0$ can be derived as follows:

$$\begin{aligned}\|x^m x^{-\mu-1/2} N_{\mu}^{-1}\varphi(x)\|_A &\leq x^m \int_x^{\infty} \|t^{-\mu-1/2}\varphi(t)\|_A\,dt \\ &\leq \int_x^{\infty} \|t^m t^{-\mu-1/2}\varphi(t)\|_A\,dt \\ &\leq \int_0^{+\infty} \left\|\frac{1}{1+t^2}(t^{m+1} + t^{m+2})t^{-\mu-3/2}\varphi(t)\right\|_A dt \\ &\leq \int_0^{\infty} \frac{1dt}{1+t^2} \cdot \sup_{t\in I} \|(t^{m+1} + t^{m+3})t^{-\mu-3/2}\varphi(t)\|_A .\end{aligned}$$

Therefore

$$\gamma^{\mu}_{m,0}(N_{\mu}^{-1}\varphi) \leq \frac{\pi}{2}[\gamma^{\mu+1}_{m+1,0}(\varphi) + \gamma^{\mu+1}_{m+3,0}(\varphi)], \qquad m = 0, 1, 2, \ldots .$$

It follows from the above that N_{μ}^{-1} is a continuous linear mapping of $H_{\mu+1}(A)$ into $H_{\mu}(A)$.

Let $\mu \in R$ and let k be a positive integer such that $\mu + k \geq -\frac{1}{2}$. Assume that $\varphi \in H_\mu(A)$ and define $h_{\mu,k}$ on $H_\mu(A)$ by (see Koh [4])

$$\Phi(x) = h_{\mu,k}(\phi(y)) \triangleq (-1)^k x^{-k} h_{\mu+k} N_{\mu+k-1} \cdots N_{\mu+1} N_\mu \varphi(y).$$

Let $\Phi(x) \in H_\mu(A)$ and define $h_{\mu,k}^{-1}$ on $H_\mu(A)$ by

$$\varphi(y) = h_{\mu,k}^{-1}(\Phi(x)) \triangleq (-1)^k N_\mu^{-1} N_{\mu+1}^{-1} \cdots N_{\mu+k-1}^{-1} h_{\mu+k} x^k \Phi(x).$$

Theorem 3.1. *$h_{\mu,k}$ is an automorphism on $H_\mu(A)$. Its inverse is $h_{\mu,k}^{-1}$ and $h_{\mu,k} = h_\mu$ if $\mu \geq -\frac{1}{2}$.*

Proof. By Lemmas 3.1 and 3.2, $\varphi \to N_\mu N_{\mu+1} \cdots N_{\mu+k-1}\varphi$ is an isomorphism from $H_\mu(A)$ onto $H_{\mu+k}(A)$.

By Lemma 2.2, $h_{\mu+k}$ is an automorphism on $H_{\mu+k}(A)$ for $\mu + k \geq -\frac{1}{2}$. It follows from $\gamma_{m,k}^{\mu}(x^{-k}\varphi) = \gamma_{m,k}^{\mu+k}(\varphi)$ that $\varphi \to x^{-k}\varphi$ is an isomorphism from $H_{\mu+k}(A)$ onto $H_\mu(A)$. Therefore $h_{\mu,k}$ is an automorphism on $H_\mu(A)$. Similarly $h_{\mu,k}^{-1}$ is an automorphism on $H_\mu(A)$ and is inverse to $h_{\mu,k}$ because $h_{\mu+k}^{-1} = h_{\mu+k}$ and the inverse of $N_{\mu+k-1} \cdots N_\mu$ is $N_\mu^{-1} \cdots N_{\mu+k-1}^{-1}$.

To prove the last statement, let $\varphi(y) \in H_\mu(A)$, $\mu \geq -\frac{1}{2}$, and consider $k = 1$;

$$\begin{aligned} h_{\mu,1}\varphi &= -x^{-1} h_{\mu+1} N_\mu \varphi = -x^{-1} \int_0^\infty y^{\mu+1/2}[D_y y^{-\mu-1/2}\varphi(y)]\sqrt{xy}\,J_{\mu+1}(xy)\,dy \\ &= -x^{-1}\sqrt{xy}\,J_{\mu+1}(xy)\varphi(y)\Big|_0^\infty + \int_0^\infty \varphi(y)\sqrt{xy}\,J_\mu(xy)\,dy. \end{aligned}$$

Since $\varphi(y)$ is of rapid descent and $\sqrt{xy}\,J_{\mu+1}(xy)$ is bounded as $y \to \infty$, while $\varphi(y) = O(y^{\mu+1/2})$ and $\sqrt{xy}\,J_{\mu+1}(xy) = O(y^{\mu+3/2})$ as $y \to 0^+$, the limit terms are zero for $\mu \geq -\frac{1}{2}$. Thus $h_{\mu,1}\varphi = h_\mu\varphi$. By induction, $h_{\mu,k} = h_\mu$ for $\mu \geq -\frac{1}{2}$.

Note that the definition of $h_{\mu,k}$ is independent of the choice of k so long as $k + \mu \geq -\frac{1}{2}$. Indeed, if $k > p \geq -\mu - \frac{1}{2}$, then $h_{\mu+p,k-p} = h_{\mu+p}$ by Theorem 3.1; hence

$$\begin{aligned} h_{\mu,k}\varphi &= (-1)^k x^{-k} h_{\mu+k} N_{\mu+k-1} \cdots N_\mu \varphi \\ &= (-1)^p x^{-p} (-1)^{k-p} x^{-(k-p)} h_{\mu+p+k-p} N_{\mu+p+k-p-1} \cdots N_{\mu+p} N_{\mu+p-1} \cdots N_\mu \varphi \\ &= (-1)^p x^{-p} h_{\mu+p,k-p} N_{\mu+p-1} \cdots N_\mu \varphi \\ &= (-1)^p x^{-p} h_{\mu+p} N_{\mu+p-1} \cdots N_\mu \varphi = h_{\mu,p}\varphi. \end{aligned}$$

Definition 3.1. Let $\mu \in R$ and k be a positive integer such that $\mu + k \geq -\frac{1}{2}$. For any $f \in [H_\mu(A); B]$, the generalized Hankel transform $h'_\mu f$ is defined by

$$\langle h'_\mu f, \varphi \rangle = \langle f, h_{\mu,k}\varphi \rangle, \qquad \varphi \in H_\mu(A).$$

By Theorem 3.1 and the fact that h'_μ is the adjoint operator of $h_{\mu,k}$ on $H_\mu(A)$, we have

Theorem 3.2. *h'_μ is an automorphism on $[H_\mu(A); B]$ for all $\mu \in R$.*

Applying operator $T \triangleq N_{\mu+k-1} \cdots N_\mu$, we have

Theorem 3.3. *Let A and B be two Banach spaces. There is a bijection from $[H_\mu(A); B]$ onto $[H_\mu; [A; B]]$ defined by*

$$\langle g, \theta\rangle a = \langle f, \theta a\rangle$$

where $a \in A$, $\theta \in H_\mu$, $g \in [H_\mu; [A; B]]$, and $f \in [H_\mu(A); B]$, $\mu \in R$.

Proof. For arbitrary $\mu \in R$, we choose a positive integer k such that $\mu + k \geq -\frac{1}{2}$. The operator T is an isomorphism from ${}_\mu D_I \odot A$ onto ${}_{\mu+k}D_I \odot A$, which is dense in $H_{\mu+k}(A)$. Also T is an isomorphism from $H_\mu(A)$ onto $H_{\mu+k}(A)$. Therefore, ${}_\mu D_I \odot A$ is dense in $H_\mu(A)$. By Theorems 2.3 and 2.4, there is a bijection from $[H_\mu(A); B]$ onto $[H_\mu; [A; B]]$ satisfying the above equation.

4. Some operational formulas

We now establish certain transformation formulas relating to the Bessel-type differential operator $M_\mu N_\mu$, which are similar to those obtained in [4], but on $H_\mu(A)$.

Lemma 4.1. *Let μ be a fixed real number and k a positive integer $\geq -\mu - \frac{1}{2}$. Then for every $\varphi \in H_\mu(A)$, $h_{\mu+1,k}(N_\mu\varphi) = -xh_{\mu,k}(\varphi)$.*

Proof. By definition

$$\begin{aligned} h_{\mu+1,k}(N_\mu\varphi) &= (-1)^k x^{-k} h_{\mu+1+k} N_{\mu+1+k} \cdots N_{\mu+1} N_\mu \varphi \\ &= -xh_{\mu,k+1}(\varphi) = -xh_{\mu,k}(\varphi). \end{aligned}$$

Turning to the linear operator M_μ, we prove that $\varphi \to M_\mu\varphi$ is a continuous linear mapping of $H_{\mu+1}(A)$ onto $H_\mu(A)$. Indeed, for $\varphi \in H_{\mu+1}(A)$ and any choice m and k

$$\begin{aligned} \gamma^\mu_{m,k}(M_\mu\varphi) &= \sup_{x\in I} \|x^m (x^{-1}D)^k x^{-2\mu-1} D x^{2\mu+2} x^{-\mu-3/2}\varphi(x)\|_A \\ &= \sup_{x\in I} \|(2\mu+2)x^m(x^{-1}D)^k x^{-\mu-3/2}\varphi(x) \\ &\qquad + x^m (x^{-1}D)^k x^2 (x^{-1}D) x^{-\mu-3/2}\varphi(x)\|_A \\ &= \cdots = \sup_{x\in I} \|2(\mu+k+1)x^m (x^{-1}D)^k x^{-\mu-3/2}\varphi(x) \\ &\qquad + x^{m+2}(x^{-1}D)^{k+1} x^{-\mu-3/2}\varphi(x)\|_A \\ &\leq 2|\mu+k+1|\gamma^{\mu+1}_{m,k}(\varphi) + \gamma^{\mu+1}_{m+2,k+1}(\varphi). \end{aligned}$$

This implies our assertion.

Lemma 4.2. *Let μ and k be as in Lemma 4.1. Then for every $\varphi \in H_{\mu+1}(A)$*

$$h_{\mu,k}(M_\mu\varphi) = xh_{\mu+1,k}(\varphi).$$

Proof. Using the relation

$$N_{\mu+k-1} \cdots N_{\mu+1} N_\mu \varphi(x) = x^{\mu+k+1/2}(x^{-1}D)^k x^{-\mu-1/2}\varphi(x)$$

we have

$$\begin{aligned} h_{\mu,k}(M_\mu\varphi) &= (-1)^k x^{-k} \int_0^\infty \sqrt{xy} J_{\mu+k}(xy) y^{\mu+k+1/2} y^2 (y^{-1}D)^{k+1} y^{-\mu-1-1/2}\varphi\, dy \\ &\quad + (-1)^k x^{-k}(2\mu+2k+2) \\ &\quad \times \int_0^\infty \sqrt{xy} J_{\mu+k}(xy) \cdot y^{\mu+k+1/2}(y^{-1}D)^k y^{-\mu-1/2-1}\varphi\, dy. \end{aligned} \tag{4.1}$$

We now show that $xh_{\mu+1,k}(\varphi)$ reduces to (4.1). Indeed

$$xh_{\mu+1,k}(\varphi) = (-1)^k x^{-k+1} \int_0^\infty \sqrt{xy}\, J_{\mu+k+1}(xy) y^{\mu+k+1+1/2} (y^{-1}D)^k y^{-\mu-3/2} \varphi\, dy .$$

From the formula (see [5])

$$J_{\mu+k+1}(xy) = -x^{-1} y^{\mu+k} D y^{-\mu-k} J_{\mu+k}(xy)$$

and an integration by parts, we obtain

$$\begin{aligned} xh_{\mu+1,k}(\varphi) &= (-1)^{k+1} x^{-k+1/2} \\ &\quad \times \int_0^\infty y^{2\mu+2k+2} (y^{-1}D)^k y^{-\mu-3/2} \varphi \cdot D[y^{-\mu-k} J_{\mu+k}(xy)]\, dy \\ &= (-1)^{k+1} x^{-k+1/2} \left\{ y^{\mu+k+2} J_{\mu+k}(xy) (y^{-1}D)^k y^{-\mu-3/2} \varphi \Big|_0^\infty \right. \\ &\qquad - \int_0^\infty y^{-\mu-k} J_{\mu+k}(xy) \\ &\qquad\qquad \left. \times D[y^{2\mu+2k+2} (y^{-1}D)^k \cdot y^{-\mu-3/2} \varphi]\, dy \right\} . \end{aligned}$$

The limit terms vanish because $\varphi \in H_{\mu+1}(A)$. Since

$$\begin{aligned} D[y^{2\mu+2k+2} (y^{-1}D)^k y^{-\mu-3/2} \varphi] &= y^{2\mu+2k+3} (y^{-1}D)^{k+1} y^{-\mu-3/2} \\ &\quad + (2\mu + 2k + 2) \cdot y^{2\mu+2k+1} (y^{-1}D) y^{-\mu-3/2} \varphi , \end{aligned}$$

we see that $xh_{\mu+1,k}(\varphi)$ equals the right-hand side of (4.1). This completes the proof.

Lemma 4.3. *Let μ be any fixed real number and k a positive integer $\geq -\mu - \frac{1}{2}$. Then, for every $\varphi \in H_\mu(A)$,*

$$h_{\mu,k}(M_\mu N_\mu \varphi) = -x^2 h_{\mu,k}(\varphi) .$$

Proof. From Lemmas 4.1 and 4.2.

Similarly, we can show

Lemma 4.4. *Let μ be any fixed real number and k a positive integer $\geq -\mu - \frac{1}{2}$. Then, for every $\varphi \in H_\mu(A)$,*

$$M_\mu N_\mu h_{\mu,k} \varphi = h_{\mu,k}(-x^2 \varphi) .$$

Theorem 4.1. *For any real μ and $f \in [H_\mu(A); B]$,*

$$M_\mu N_\mu h'_\mu f = h'_\mu[-x^2 f] .$$

Proof. It follows from Lemma 4.3 that

$$\begin{aligned} \langle h'_\mu[-x^2 f], \varphi \rangle &= \langle -x^2 f, h_{\mu,k} \varphi \rangle = \langle f, -x^2 h_{\mu,k} \varphi \rangle = \langle f, h_{\mu,k}(M_\mu N_\mu \varphi) \rangle \\ &= \langle h'_\mu f, M_\mu N_\mu \varphi \rangle = \langle M_\mu N_\mu h'_\mu, \varphi \rangle . \end{aligned}$$

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