THE HAYMAN-WU CONSTANT

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Abstract. The Hayman-Wu constant is at least $\pi^2$.

Let $D$ be the open unit disc and $T$ its boundary. The length of a curve $K$ is denoted $|K|$. The Hayman-Wu theorem says that there is a constant $C$ such that if $f(z)$ is univalent in $D$ and $L$ is any line then $|f^{-1}(L)| \leq C$ (see [3]). The Hayman-Wu constant is the least possible value of $C$. Its numerical value is unknown, but in [4] it is proved that $C \leq 4\pi$. It has been conjectured that $C = 8 \int_0^1 dx/\sqrt{1 + x^4}$ (see [1]); however, we will prove

Theorem. $C \geq \pi^2$.

Flinn proved in [2] that if $f(D)$ contains one component of $\mathbb{C}\setminus L$ then $|f^{-1}(L)| \leq \pi^2$. Our example shows that this is the best possible result in this case. The proof uses an elementary fact about harmonic measure: If $I$ is a subarc of $T$ and $0 < c < 1$ then the level curve $\omega(z, I, D) = c$ is a circular arc through the endpoints of $I$ meeting $T\setminus I$ at an angle of $c\pi$.

Let $\Pi^+$ and $\Pi^-$ be the upper and lower half planes respectively. If $I$ is an interval of the real line and $0 < \varepsilon < 1$ then let $C_{I,\varepsilon}$ be the circle centered in $\Pi^+$ meeting $\mathbb{R}$ at the endpoints of $I$ such that the (least) angle between $C_{I,\varepsilon}$ and $\mathbb{R}$ is $\varepsilon$. We define $C_{I,\varepsilon} \cap \Pi^+ = S_{I,\varepsilon}$. Let $\Omega_{I,\varepsilon}$ be the unbounded component of $\mathbb{C}\setminus (S_{I,\varepsilon} \cup S_{I,\varepsilon/2})$. Two lemmas are needed.

Lemma 1. For $z \in I$, $\omega(z) = \omega(z, S_{I,\varepsilon}, \Omega_{I,\varepsilon}) < \frac{1}{2} + \varepsilon$.

Proof. Without loss of generality $I$ equals $[0, 1]$. If we use the transformation $g(z) = 1/z - 1$, we may assume that $\Omega_{I,\varepsilon} = \{ re^{i\phi} : r > 0, -\pi + \varepsilon < \phi < \pi + \varepsilon/2\}$ and that $I = \mathbb{R}^+$. Then $\omega(z)$ is given by the formula

$$\omega(re^{i\phi}) = (\pi + \varepsilon/2 - \phi)/(2\pi - \varepsilon/2).$$

Therefore, $\omega(z) = (\pi + \varepsilon/2)/(2\pi - \varepsilon/2) < \frac{1}{2} + \varepsilon$ for $z \in \mathbb{R}^+$.

Lemma 2. For every $\delta > 0$ there exist numbers $b > 0$ and $\varepsilon > 0$ such that if $I$ is a subarc of $T$ of length less than $b$ and $K$ is a crosscut in $D$ connecting

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the endpoints of \( I \) satisfying \( \omega(z, I, D) < \frac{1}{2} + \varepsilon \) for every \( z \in K \), then \( |K| > |I|(1 - \delta)\pi/2 \).

**Proof.** \( K \) lies outside the convex curve \( \omega(z, I, D) = \frac{1}{2} + \varepsilon \). If \( |I| \) and \( \varepsilon \) are small then this curve is almost a half circle whose diameter is almost \( |I| \). A routine but tedious calculation shows that

\[
|\omega(z, I, D)| = \frac{1}{2} + \varepsilon > (\sin(|I|/2))(\pi - |I| - 2\varepsilon \pi).
\]

**Proof of the theorem.** If \( \delta > 0 \) choose \( \varepsilon \) as in Lemma 2. Define \( I_0^1 = [0, 1] \) and \( d = \text{diam}(C_{I_0^1, \varepsilon/2}) \). For \( k \in \mathbb{Z} \) let \( I_k^1 = I_0^1 + 2kd \). The circles \( C_{I_k^1, \varepsilon/2} \) are disjoint. Let \( \mathbb{R} \setminus \bigcup I_k^1 = \bigcup J_k^1 \), where the intervals \( J_k^1 \) are disjoint. Choose closed intervals \( I_k^2 \subset \bigcup J_k^1 \) such that:

(i) \( S_{I_k^1, \varepsilon/2} \cap S_{I_n^1, \varepsilon/2} = \emptyset \) for \( m \neq n \);

(ii) \( S_{I_m^1, \varepsilon/2} \cap S_{I_n^1, \varepsilon/2} = \emptyset \) for all \( m, n \);

(iii) Each compact subset of \( C \) intersects only finitely many \( J_k^1 \);

(iv) \( |\bigcup I_k^2 \cap J_n^1| > |J_n^1|/3d \) for all \( m \).

We can obtain (iv) by choosing each \( I_k^2 \) small. Let \( \mathbb{R} \setminus \bigcup (J_m^2 \cup I_n^2) = \bigcup J_m^2 \). Continue the construction inductively.

Renumber the set \( \{I_n^2\} = \{I_n\} \). Define \( S_n = S_{I_n, \varepsilon} \) and let \( O_n \) be the inside of \( C_{I_n, \varepsilon} \). Define \( \Omega = (\bigcup O_n) \cup \partial \Omega^- \). The domain \( \Omega \) is simply connected and the boundary of \( \Omega \) equals \( (\bigcup S_n) \cup E \) where \( E \subset \mathbb{R} \). This is a Jordan arc, which is locally rectifiable since \( |S_n|/|I_n| = \text{constant} \). Therefore \( \omega(z, E, \Omega) \equiv 0 \) since \( |E| = 0 \) by (iv). It follows that if \( f(z) \) maps \( D \) conformally onto \( \Omega \) then

\[ \sum |f^{-1}(S_n)| = 2\pi. \]

By comparison \( \omega_n(z) = \omega(z, S_n, \Omega) < \omega(z, S_n, \Omega_{I_n, \varepsilon}) \). Therefore, by Lemma 1, \( \omega_n(z) < \frac{1}{2} + \varepsilon \) for \( z \in I_n \). Choose \( f(z) \) such that \( f(0) = -ia \) where \( a \) is so large that \( \omega_n(-ia) < b \) for all \( n \). The constant \( b \) comes from Lemma 2. \( f^{-1}(I_n) \) is a crosscut in \( D \) connecting the endpoints of \( f^{-1}(S_n) \). Lemma 2 shows that \( |f^{-1}(I_n)| > |f^{-1}(S_n)|(1 - \delta)\pi/2 \). This proves the theorem since

\[
|f^{-1}(\mathbb{R})| = \sum |f^{-1}(I_n)| \geq \sum |f^{-1}(S_n)|(1 - \delta)\pi/2 = \pi^2(1 - \delta).
\]

**Conjecture.** \( C = \pi^2 \).

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**References**


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