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THE HAYMAN-WU CONSTANT

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ABSTRACT. The Hayman-Wu constant is at least π^2.

Let D be the open unit disc and T its boundary. The length of a curve K is denoted $|K|$. The Hayman-Wu theorem says that there is a constant C such that if $f(z)$ is univalent in D and L is any line then $|f^{-1}(L)| \leq C$ (see [3]). The Hayman-Wu constant is the least possible value of C. Its numerical value is unknown, but in [4] it is proved that $C \leq 4\pi$. It has been conjectured that $C = 8\int_0^1 dx/\sqrt{1+x^4}$ (see [1]); however, we will prove

Theorem. $C \geq \pi^2$.

Flinn proved in [2] that if $f(D)$ contains one component of $\mathbb{C}\backslash L$ then $|f^{-1}(L)| \leq \pi^2$. Our example shows that this is the best possible result in this case. The proof uses an elementary fact about harmonic measure: If I is a subarc of T and $0 < c < 1$ then the level curve $\omega(z, I, D) = c$ is a circular arc through the endpoints of I meeting $T\backslash I$ at an angle of $c\pi$.

Let Π^+ and Π^- be the upper and lower half planes respectively. If I is an interval of the real line and $0 < \varepsilon < 1$ then let $C_{I,\varepsilon}$ be the circle centered in Π^+ meeting $\mathbb{R}$ at the endpoints of I such that the (least) angle between $C_{I,\varepsilon}$ and $\mathbb{R}$ is ε. We define $C_{I,\varepsilon} \cap \Pi^+ = S_{I,\varepsilon}$. Let $\Omega_{I,\varepsilon}$ be the unbounded component of $\mathbb{C}\backslash\overline{(S_{I,\varepsilon} \cup S_{I,\varepsilon/2})}$. Two lemmas are needed.

Lemma 1. *For* $z \in I$, $\omega(z) = \omega(z, S_{I,\varepsilon}, \Omega_{I,\varepsilon}) < \frac{1}{2} + \varepsilon$.

Proof. Without loss of generality I equals $[0, 1]$. If we use the transformation $g(z) = 1/z - 1$, we may assume that $\Omega_{I,\varepsilon} = \{re^{i\phi}: r > 0, -\pi+\varepsilon < \phi < \pi+\varepsilon/2\}$ and that $I = \mathbb{R}^+$. Then $\omega(z)$ is given by the formula

$$\omega(re^{i\phi}) = (\pi + \varepsilon/2 - \phi)/(2\pi - \varepsilon/2).$$

Therefore, $\omega(z) = (\pi + \varepsilon/2)/(2\pi - \varepsilon/2) < \frac{1}{2} + \varepsilon$ for $z \in \mathbb{R}^+$.

Lemma 2. *For every* $\delta > 0$ *there exist numbers* $b > 0$ *and* $\varepsilon > 0$ *such that if* I *is a subarc of* T *of length less than* b *and* K *is a crosscut in* D *connecting*

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the endpoints of I *satisfying* $\omega(z, I, D) < \frac{1}{2} + \varepsilon$ *for every* $z \in K$, *then* $|K| > |I|(1-\delta)\pi/2$.

Proof. K lies outside the convex curve $\omega(z, I, D) = \frac{1}{2} + \varepsilon$. If $|I|$ and ε are small then this curve is almost a half circle whose diameter is almost $|I|$. A routine but tedious calculation shows that

$$|\omega(z, I, D) = \tfrac{1}{2} + \varepsilon| > (\sin(|I|/2))(\pi - |I| - 2\varepsilon\pi).$$

Proof of the theorem. If $\delta > 0$ choose ε as in Lemma 2. Define $I_0^1 = [0, 1]$ and $d = \operatorname{diam}(C_{I_0^1, \varepsilon/2})$. For $k \in Z$ let $I_k^1 = I_0^1 + 2kd$. The circles $C_{I_k^1, \varepsilon/2}$ are disjoint. Let $\mathbb{R} \setminus \bigcup I_k^1 = \bigcup J_m^1$, where the intervals J_k^1 are disjoint. Choose closed intervals $I_n^2 \subset \bigcup J_k^1$ such that:

(i) $S_{I_m^2, \varepsilon/2} \cap S_{I_n^2, \varepsilon/2} = \varnothing$ for $m \neq n$;
(ii) $S_{I_m^2, \varepsilon/2} \cap S_{I_n^1, \varepsilon/2} = \varnothing$ for all m, n;
(iii) Each compact subset of $\mathbb{C}$ intersects only finitely many I_k^2;
(iv) $|\bigcup I_k^2 \cap J_m^1| > |J_m^1|/3d$ for all m.

We can obtain (iv) by choosing each I_k^2 small. Let $\mathbb{R} \setminus (\bigcup I_m^2 \cup I_n^1) = \bigcup J_m^2$. Continue the construction inductively.

Renumber the set $\{I_m^k\} = \{I_n\}$. Define $S_n = S_{I_n, \varepsilon}$ and let O_n be the inside of $C_{I_n, \varepsilon}$. Define $\Omega = (\bigcup O_n) \cup \Pi^-$. The domain Ω is simply connected and the boundary of Ω equals $(\bigcup S_n) \cup E$ where $E \subset \mathbb{R}$. This is a Jordan arc, which is locally rectifiable since $|S_n|/|I_n| = \text{constant}$. Therefore $\omega(z, E, \Omega) \equiv 0$ since $|E| = 0$ by (iv). It follows that if $f(z)$ maps D conformally onto Ω then $\sum |f^{-1}(S_n)| = 2\pi$.

By comparison $\omega_n(z) = \omega(z, S_n, \Omega) < \omega(z, S_n, \Omega_{I_n, \varepsilon})$. Therefore, by Lemma 1, $\omega_n(z) < \frac{1}{2} + \varepsilon$ for $z \in I_n$. Choose $f(z)$ such that $f(0) = -ia$ where a is so large that $\omega_n(-ia) < b$ for all n. The constant b comes from Lemma 2. $f^{-1}(I_n)$ is a crosscut in D connecting the endpoints of $f^{-1}(S_n)$. Lemma 2 shows that $|f^{-1}(I_n)| > |f^{-1}(S_n)|(1-\delta)\pi/2$. This proves the theorem since

$$|f^{-1}(\mathbb{R})| = \sum |f^{-1}(I_n)| \geq \sum |f^{-1}(S_n)|(1-\delta)\pi/2 = \pi^2(1-\delta).$$

Conjecture. $C = \pi^2$.

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