

THE STABLE NEIGHBORHOOD THEOREM AND LENGTHS OF CLOSED GEODESICS

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ABSTRACT. We derive a generalized collar lemma, called the stable neighborhood theorem, for nonsimple closed geodesics. As an application, we show that there is a lower bound for the length of a closed geodesic having crossing number k on a hyperbolic surface. This lower bound only depends on k and tends to infinity as k goes to infinity. Also, we show that the shortest nonsimple closed geodesic on a closed hyperbolic surface has (geometric) crossing number bounded above by a constant which only depends on the genus.

0. INTRODUCTION

It is well known that a simple closed geodesic on a hyperbolic surface has a collar neighborhood whose width only depends on the length of the geodesic [Bu, Ha, K, M, Ma, R]. In this paper, we derive a type of generalized collar lemma (the stable neighborhood theorem) for closed geodesics with self-intersection. The notion of a collar is replaced by the concept of a stable neighborhood about a closed geodesic. The stable neighborhood theorem says that a closed geodesic on a hyperbolic surface has a stable neighborhood whose width only depends on the length of the closed geodesic.

Hempel, Nakanishi, and Yamada [He, N, Y1, Y2] have shown that there is a universal lower bound for the length of a nonsimple closed geodesic on a hyperbolic surface. As a consequence of the stable neighborhood theorem, we show that in fact one can improve this lower bound (Corollary 1.2) by knowing the crossing number of the closed geodesic.

Finally, using a theorem of Bers on pants decompositions of Riemann surfaces, we show that the number of crossings of the shortest closed geodesic with at least k -fold crossings on a closed hyperbolic surface is bounded from above by a constant depending only on k and the genus of the surface.

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1. STATEMENT OF RESULTS

The *neighborhood of width d* , $U_d(\omega)$, about a geodesic ω in a hyperbolic surface is the set of all points within a distance d from the geodesic. Suppose ω is a closed geodesic. Then the neighborhood $U_d(\omega)$ is said to be *stable* if for any two connected (smooth) lifts of ω , say ω_1 and ω_2 , we have

$$\omega_1 \cap \omega_2 \neq \emptyset \quad \text{if and only if} \quad U_d(\omega_1) \cap U_d(\omega_2) \neq \emptyset.$$

Clearly a closed geodesic has stable neighborhoods for sufficiently small widths (which depend on the hyperbolic structure of the surface). Our interest is in finding stable neighborhoods whose widths depend only on the length of the geodesic and not on the underlying hyperbolic structure of the surface.

We associate to each closed geodesic ω of length $\ell = \ell(\omega)$ the function,

$$c(\omega) = c(\ell) = \log \coth(\ell/4).$$

Theorem 1.1 (The stable neighborhood theorem). *A closed geodesic of length ℓ on a hyperbolic surface has a stable neighborhood of width $c(\ell)$. Furthermore, two disjoint closed geodesics ω_1 and ω_2 on the surface having lengths ℓ_1 and ℓ_2 have disjoint stable neighborhoods of widths $c(\ell_1)$ and $c(\ell_2)$, respectively, if ω_1 and ω_2 are separated by a disjoint union of simple closed geodesics.*

We remark that the separation condition in the first part of the theorem is necessary. For example, a pair of pants has closed geodesics that get arbitrarily close to any fixed boundary geodesic. Of course, the crossing numbers of these closed geodesics get arbitrarily large as they get closer to the boundary geodesic. Thus the width of the stable neighborhood must take into account the crossing number of the other closed geodesic.

Corollary 1.2. *There exists an increasing sequence M_k (for $k = 1, 2, 3, \dots$) tending to infinity so that if ω is a closed geodesic with self-intersection number k , then $\ell(\omega) > M_k$. Thus the length of a closed geodesic gets arbitrarily large as its self-intersection gets large.*

It is well known that the shortest closed geodesic on a hyperbolic surface which is not a thrice punctured sphere is simple. In other words, it has crossing number zero. Furthermore, the length of the shortest closed geodesic on a hyperbolic surface of genus g is bounded from above by a constant which only depends on the genus of the surface (see [Ber]). Following these lines, we let $s_k(S)$ be the length of the shortest closed geodesic with at least k -fold intersection on the hyperbolic surface S . We are interested in upper bounds (depending only on topological data) for the lengths of these short geodesics.

Proposition 1.3. *There are constants $N_k(g)$, depending only on the crossing number of the geodesic and the genus of the surface, so that the shortest closed geodesic with at least k -fold intersection on a surface of genus g has length bounded from above by $N_k(g)$.*

Peter Buser has asked whether the shortest nonsimple closed geodesic on a hyperbolic surface has only one self-intersection. As an elementary consequence of Corollary 1.2 and Proposition 1.3, we have

Corollary 1.4. *Suppose γ_S is the shortest closed geodesic with crossing number at least k on the hyperbolic surface S of genus g . Then the crossing number*

of γ_S is bounded from above by a constant which only depends on k and g . Of course, for $k = 0$ we know that the upper bound can be taken to be zero.

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2. BASICS AND NOTATION

The reader is referred to the paper [Ba] for most of the basic definitions involved. In this paper, we use $d(\cdot, \cdot)$ to denote hyperbolic distance in the hyperbolic plane.

In order to prove the above theorems we need to define some words. A *fuchsian* group is a discrete subgroup of orientation preserving isometries of the hyperbolic plane \mathbb{H}^2 . The quotient of the hyperbolic plane by a fuchsian group is a *hyperbolic surface*. A hyperbolic surface (and the group representing it) is said to be a *pair of pants* if it is topologically a thrice-punctured sphere.

Let \mathbb{T}^1 be the unit circle and suppose $\omega : \mathbb{T}^1 \rightarrow S$ is a (parametrized) closed geodesic on the hyperbolic surface S . Define the *crossing* (or *self-intersection*) *number* of ω to be the number of transverse intersections that ω makes with itself.

The action of the isometries on the hyperbolic plane extends to the circle at infinity. An isometry having two fixed points on the circle at infinity is known as a hyperbolic element. The *axis* of a hyperbolic element is the unique geodesic in the hyperbolic plane joining the fixed points. We use $A(\gamma)$ to denote the axis of γ . The *translation length* of γ , denoted $\ell(\gamma)$, is the hyperbolic distance that a point on $A(\gamma)$ is moved. We will use the same notation for the length of a closed geodesic on the surface.

Suppose γ and β are hyperbolic elements having disjoint axes. We say that γ and β are *standard generators* for a pair of pants G if

- (1) γ and β generate a torsion-free fuchsian group which represents a pair of pants;
- (2) after possibly changing the orientations of γ and β so that their axes are oriented to the right of each other, γ , β , and $\beta\gamma$ are primitive boundary elements of G , each representing different boundary components of the pair of pants.

The main tool in the following arguments is the pair of pants theorem [Ba], which states that γ and β are standard generators for a pair of pants if and only if $c(\gamma) + c(\beta) \leq d$, where d is the distance from the axis of γ to the axis of β .

3. THE PROOFS OF THEOREM 1.1 AND COROLLARY 1.2

In order to prove Theorem 1.1 we need the following lemma, which appears in the proof of a proposition in [GM, p. 19].

Lemma 3.1. *Two hyperbolic elements with equal translation lengths having disjoint axes in a torsion-free fuchsian group form standard generators for a pair of pants.*

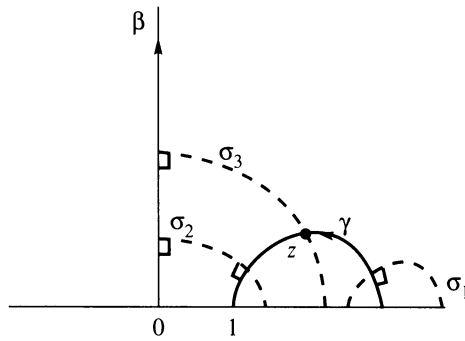


FIGURE 1

For the reader’s convenience, we supply the proof.

Proof. Without loss of generality, we can normalize the hyperbolic elements and call them β and γ , as in Figure 1. Now, as usual, embed the group $\langle \beta, \gamma \rangle$ as an index two subgroup of the reflection group $\langle \sigma_1, \sigma_2, \sigma_3 \rangle$. Thus $\gamma = \sigma_2\sigma_1$ and $\beta = \sigma_3\sigma_2$. The pair (γ, β) will be standard generators for a pair of pants if and only if the σ_i -reflection circles bound a free polygon (see [Ba] for the details).

If $\beta A(\gamma) \cap A(\gamma) \neq \emptyset$, this means that the σ_3 -reflection circle must intersect $A(\gamma)$. Then either the σ_3 -reflection separates σ_2 from σ_1 or the σ_1 reflection circle intersects $A(\beta)$. Note that the reflection circles cannot intersect since this would introduce torsion into the original group. Let z be the intersection point between $A(\gamma)$ and the σ_3 -reflection circle. Now, if the σ_3 -reflection circle separates then (Figure 1)

$$\frac{\ell(\beta)}{2} = d(\sigma_2, \sigma_3) < d(\sigma_2, z) < d(\sigma_2, \sigma_1) = \frac{\ell(\gamma)}{2},$$

where the first inequality follows from the fact that the common orthogonal between two geodesics realizes the minimum distance between any two points on the geodesics. This clearly contradicts our assumption that $\ell(\gamma) = \ell(\beta)$. If, on the other hand, the σ_1 -reflection circle intersects $A(\beta)$ then the same argument as above works by considering the distance $\frac{\ell(\gamma)}{2} = d(\sigma_1, \sigma_2)$. Thus the σ_i -reflection circles must bound a free polygon and hence are standard generators for a pair of pants. \square

Suppose ω is a closed geodesic on the hyperbolic surface S . Fix a lift β of the geodesic ω . Consider the region \mathcal{R} of points in the hyperbolic plane that are a distance less than $c(\omega)$ from the axis $A(\beta)$. That is,

$$\mathcal{R} = \{z \in \mathbb{H}^2 \mid d(z, A(\beta)) < c(\omega)\}.$$

We claim that this projects to S as a stable neighborhood of ω .

Proof of Theorem 1.1. Let G be the fuchsian group representing the hyperbolic surface S . We first would like to show that ω has a stable neighborhood of width $c(\omega)$. Consider the region \mathcal{R} constructed above and suppose that

$g\mathcal{R} \cap \mathcal{R} \neq \emptyset$ for some $g \in G$. We claim that $gA(\beta) \cap A(\beta) \neq \emptyset$. To see this, consider the hyperbolic elements β and $g\beta g^{-1}$ and suppose that the g -translate of $A(\beta)$ does not cross $A(\beta)$. Then since the translation length of β equals the translation length of its conjugate $g\beta g^{-1}$, by Lemma 3.1, $(\beta, g\beta g^{-1})$ are standard generators for a pair of pants. Hence

$$2c(\omega) = c(\beta) + c(g\beta g^{-1}) \leq d(A(\beta), A(g\beta g^{-1})),$$

contradicting our assumption that $g\mathcal{R} \cap \mathcal{R} \neq \emptyset$. Thus, $gA(\beta) \cap A(\beta) \neq \emptyset$. This implies that the region \mathcal{R} projects to the desired stable neighborhood.

Now, suppose ω_1 and ω_2 are disjoint closed geodesics which are separated by simple closed geodesics. If the stable neighborhoods of these geodesics overlap, then we can find lifts γ and β to the fuchsian group G so that the length d of the common orthogonal between them satisfies $d < c(\gamma) + c(\beta)$. On the other hand, since the orthogonal from the axis $A(\gamma)$ to the axis $A(\beta)$ must intersect the axis of a simple hyperbolic in G , it must be that $\beta A(\gamma) \cap A(\gamma) = \emptyset$ and $\gamma A(\beta) \cap A(\beta) = \emptyset$. This implies that (γ, β) are standard generators for a pair of pants and hence, by the pair of pants theorem, $c(\gamma) + c(\beta) \leq d$. This is clearly a contradiction. \square

Before we prove Corollary 1.2, we need a couple of lemmas.

Lemma 3.2. *There exists a decreasing function $A = A(\ell)$ depending only on ℓ and tending to zero so that if ω is any closed geodesic on a hyperbolic surface of length ℓ , then any geodesic loop contained in ω has length bigger than A . In other words, any segment in ω of length less than or equal to $A(\ell)$ is embedded.*

Proof. Suppose ω_1 is a geodesic loop in ω , and let ω_2 be the other geodesic loop so that $\omega = \omega_1 \cup \omega_2$. Using Theorem (8.3.1) of [Bea] we have

$$\ell(\omega_1) \geq 2 \sinh^{-1} \left(\frac{1}{\sinh \ell(\omega_2)/2} \right) > 2 \sinh^{-1} \left(\frac{1}{\sinh \ell/2} \right),$$

where the last inequality follows from the fact that ω_2 is shorter than ω . Setting A to be this last term yields the lemma. \square

Lemma 3.3. *Suppose β is a closed geodesic of length ℓ on a hyperbolic surface S having at least one self-intersection. There exists a decreasing function $B = B(\ell)$, depending only on ℓ and tending to zero so that if I and J are any two embedded compact segments in β having disjoint interiors and common endpoints, then either*

$$\ell(I) > B \quad \text{or} \quad \ell(J) > B.$$

Proof. Set $S = \mathbb{H}^2/G$ and label the common endpoints z and w . Observe that I and J cannot meet smoothly at both z and w , since that would force the geodesic ω to be simple. Also, if I and J meet smoothly at one point and not the other then $I \cup J$ is a loop in ω and hence by Lemma 3.2 either $\ell(I) > A/2$ or $\ell(J) > A/2$. Thus set B to be $A/2$ in this case.

From now on assume that the intersections at z and w are not smooth. Then in \mathbb{H}^2 , there exists a triple of distinct geodesics (L_1, L_2, L_3) each one the

axis of a hyperbolic element in G representing β (these elements are conjugate in G) satisfying:

- (1) $L_1 \cap L_2 = z_1$ is a lift of z .
- (2) $L_2 \cap L_3 = w_1$ is a lift of w with the line segment from z_1 to w_1 covering J .
- (3) There exists a point $w_2 \in L_1$ so that the line segment from z_1 to w_2 covers I .
- (4) There exists $g \in G$ taking L_3 to L_1 with $g(w_1) = w_2$.

Now if $L_1 \cap L_3 = \emptyset$, then we have

$$\ell(J) = d(z_1, w_1) \geq s \geq 2c(\ell),$$

where s is the orthogonal distance from L_2 to L_3 and the last inequality follows from the stable neighborhood theorem.

Next if $L_1 \cap L_3 \neq \emptyset$, let x be this intersection point. There are two cases to consider.

The first case is if the distances $d(x, z_1)$ and $d(x, w_1)$ are both less than or equal to $A/4$, where A is the constant in Lemma 3.2. Since the segment from x to $g(x)$ forms a loop in β by Lemma 3.2, we have

$$\begin{aligned} A \leq d(x, g(x)) &\leq d(x, z_1) + d(z_1, g(w_1)) + d(g(w_1), g(x)) \\ &\leq A/4 + A/4 + d(z_1, g(w_1)). \end{aligned}$$

Hence, we can conclude that $\ell(I) \geq A/2$.

The other case to consider is that either the distance $d(x, z_1) > A/4$ or $d(x, w_1) > A/4$. First, observe that Theorem 11.6.8 of [Bea] says that there is a positive lower bound to the angle of intersection at x between L_1 and L_3 ; moreover, this lower angle bound only depends on ℓ , the length of β . Now in either case, using the appropriate hyperbolic sine rule coupled with our lower angle bound yields a lower bound $D = D(\ell)$ for either $\ell(I)$ or $\ell(J)$.

To finish the proof, set $B = \min(A/2, 2c(\ell), D)$ and note that B is decreasing since $A(\ell)$, $c(\ell)$, and $D(\ell)$ are decreasing functions of ℓ . \square

Proof of Corollary 1.2. Suppose ω has crossing number k . Cut ω into n segments each of length $B(\ell)/2$ except possibly for one segment whose length is smaller than $B(\ell)/2$. In any event,

$$(3.4) \quad n \leq 2 \frac{\ell}{B(\ell)/2}.$$

If $k < n$, then applying inequality (3.4) we get that

$$(3.5) \quad \ell > \left(\frac{k}{4}\right) B(\ell).$$

Next suppose $k \geq n$. Then there must be a segment containing k/n crossings. Since, by Lemma 3.3, each crossing in this segment contributes at least $B(\ell)/2$, we have

$$\ell > \frac{k}{n} \left(\frac{B(\ell)}{2}\right).$$

Using inequality (3.4) and solving the above inequality for ℓ we have

$$(3.6) \quad \ell > \sqrt{\frac{k}{2}} \left(\frac{B(\ell)}{2} \right).$$

Finally, let x_k and y_k be the unique fixed points of the right-hand sides of (3.5) and (3.6), respectively. Setting $M_k = \min(x_k, y_k)$, it is easy to see that this sequence is increasing and tends to infinity with k . \square

4. THE PROOFS OF PROPOSITION 1.3 AND COROLLARY 1.4

The space \mathcal{M} of all hyperbolic metrics (structures) on a sphere with three holes can be parametrized by three nonnegative real numbers which describe the lengths of the three ordered boundary geodesics (see Abikoff [A] for the particulars). If ω is a free homotopy class of a closed curve on a hyperbolic surface S , we will denote the length of the geodesic in the free homotopy class of ω with respect to the S -metric by $\ell_S(\omega)$. We will say that a pair of pants is ε -thin if the lengths of all three of its boundary geodesics are at most ε . We first need the following lemma about ε -thin pants.

Lemma 4.1. *Let ω be a closed curve on a sphere with three holes. Then the geodesic length function $\ell_P(\omega)$ is a continuous function of \mathcal{M} and, furthermore, given $\varepsilon > 0$ there is a constant $B(\varepsilon, \omega)$, depending only on ε and the free homotopy class of ω , so that*

$$\ell_P(\omega) \leq B(\varepsilon, \omega)$$

for all ε -thin pairs of pants P .

Proof. Let $\gamma_1, \gamma_2,$ and γ_3 be the boundary curves on a sphere with three holes. The space of hyperbolic structures \mathcal{M} (note that we include punctures) can be parametrized by

$$\mathcal{M} = \{(\ell_1, \ell_2, \ell_3) \in \mathbb{R}_+^3 \cup \{0\} \mid \ell(\gamma_1) = \ell_1, \ell(\gamma_2) = \ell_2, \ell(\gamma_3) = \ell_3\}.$$

It is well known that the geodesic length function $\ell_P(\omega)$ is a continuous function on \mathcal{M} . The set of ε -thin pairs of pants form the subset

$$T_\varepsilon = \{(\ell_1, \ell_2, \ell_3) \in \mathbb{R}_+^3 \cup \{0\} \mid \ell_1, \ell_2, \ell_3 \leq \varepsilon\}.$$

T_ε is clearly compact in \mathcal{M} , and hence $\ell_P(\omega)$ has an upper bound as desired. \square

Proof of Proposition 1.3. Suppose S is a closed surface. Bers [Ber] showed that there exists a constant depending only on the genus, so that any surface of genus g has a pants decomposition where the lengths of all the boundary geodesics of the pairs of pants are bounded by this constant. We call this constant the *Bers constant* and denote it by L_g .

Using the Bers constant, it is elementary to see that there is an upper bound on $s_k(S)$, depending only on the genus of the surface; namely, for each hyperbolic surface there is an embedded L_g -thin pair of pants. Pick a free homotopy class ω having k -fold intersection number in a sphere with three holes and observe that its length on an L_g -thin pair of pants, by Lemma 4.1, is bounded by a constant depending only on L_g and, hence, only on the genus of the surface. \square

Proof of Corollary 1.4. Suppose γ_S is the shortest closed geodesic with intersection number at least k on the hyperbolic surface S of genus g . We have seen, by Proposition 1.3, that there is an upper bound $N_k(g)$ so that

$$(*) \quad \ell_S(\gamma_S) \leq N_k(g).$$

Now, we also know from Corollary 1.2 that the length of γ_S gets arbitrarily large with its self-intersection number. Putting this fact together with inequality (*) tells us that there is an upper bound, which only depends on k and g , to the number of self-intersections of γ_S . In fact, the number

$$n(k, g) = \inf\{j \in \mathbb{Z}^+ : M_j > N_k(g)\}$$

serves as an upper bound, where the M_j 's are the constants in Corollary 1.2. \square

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