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ON GROUPS RELATED TO THE HECKE GROUPS

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ABSTRACT. Let $\left[\begin{smallmatrix} 1 & \lambda_1 \\ 0 & 1 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 1 & 0 \\ \lambda_2 & 1 \end{smallmatrix}\right]$ be parabolic elements of $\mathrm{PSL}(2, R)$, where $\lambda_1, \lambda_2 > 0$. The principal result shown here is that $K(\lambda_1, \lambda_2)$, the group generated by these elements, is discrete if and only if $\lambda_1\lambda_2 \geq 4$, or $\lambda_1\lambda_2 = 4\cos^2(\pi/p)$, where p is an integer ≥ 3. When $\lambda_1\lambda_2 = 4\cos^2(\pi/p)$, $K(\lambda_1, \lambda_2)$ is conjugate to the classical Hecke group $H(2\cos(\pi/p))$ if p is odd; while if p is even, $K(\lambda_1, \lambda_2)$ is conjugate to a subgroup of $H(2\cos(\pi/p))$ of index 2. When $\lambda_1\lambda_2 \geq 4$, $K(\lambda_1, \lambda_2)$ is conjugate to a subgroup of $H(\sqrt{(\lambda_1\lambda_2)})$ of index 2. In all of these cases $K(\lambda_1, \lambda_2)$ is the free product of two cyclic groups.

This article has its genesis in [3], a continuing investigation of some little-studied aspects of the Hecke correspondence between modular forms, on the one hand, and Dirichlet series with functional equations, on the other. To describe the questions treated in [3], consider the ordinary Dirichlet series

$$\phi(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \qquad \psi(s) = \sum_{n=1}^{\infty} b_n n^{-s},$$

where the coefficients have at most polynomial growth in n, so that both series converge in a right half-plane. Assume as well that $\phi(s)$ and $\psi(s)$ can be continued meromorphically into the entire s-plane and satisfy a general functional equation of the "Hecke type",

$$(1) \qquad (2\pi/\lambda_1)^{-s}\Gamma(s)\phi(s) = C(2\pi/\lambda_2)^{-(k-s)}\Gamma(k-s)\psi(k-s);$$

here $\Gamma(s)$ is the gamma-function, k is a rational integer, λ_1 and λ_2 are positive numbers, and C is complex. (See [1, 2, 4].)

The questions considered in [3] include: (i) which pairs λ_1, λ_2 admit nontrivial solutions $\phi(s)$, $\psi(s)$ of (1)?; (ii) with fixed λ_1 and λ_2, to what extent can $\phi(s)$ and $\psi(s)$ differ? An illustration of the results contained in [3] is: if $\lambda_1 = \lambda_2 = 1$ in (1), then $\phi(s) = C\psi(s)$. As one might guess, the results of [3] arise from a consideration of the inverse Mellin transforms of $(2\pi/\lambda_1)^{-s}\Gamma(s)\phi(s)$ and $(2\pi/\lambda_2)^{-s}\Gamma(s)\psi(s)$. This brings into play modular functions with respect to certain groups of real linear fractional transformations acting on the upper half-plane. In [3] the problem arises naturally to determine when these groups are discrete; this will be the problem treated here.

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We now describe these groups. Set

$$T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad S(\lambda) = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}, \qquad W(\lambda) = \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix},$$

regarded as elements of $\mathrm{PSL}(2, R)$. Suppose that $\lambda_1, \lambda_2 > 0$, and put

$$K(\lambda_1, \lambda_2) = \{S(\lambda_1), W(\lambda_2)\},$$

the group generated by $S(\lambda_1) = \left[\begin{smallmatrix} 1 & \lambda_1 \\ 0 & 1 \end{smallmatrix}\right]$, and $W(\lambda_2) = \left[\begin{smallmatrix} 1 & 0 \\ \lambda_2 & 1 \end{smallmatrix}\right]$. We also set

$$K(\lambda) = \{S(\lambda), W(\lambda)\}, \qquad H(\lambda) = \{T, S(\lambda)\}.$$

Then $H(\lambda)$ is a Hecke group, $K(\lambda)$ is a subgroup of $H(\lambda)$, and $K(\lambda) = K(\lambda, \lambda)$.

Although we are primarily interested in $K(\lambda_1, \lambda_2)$, we will work instead with the group $K(\lambda)$ and then carry over the results to $K(\lambda_1, \lambda_2)$. This procedure is justified by the following lemma:

Lemma 1. *Suppose that* $\lambda_1, \lambda_2, \mu_1, \mu_2 > 0$, $\lambda_1\lambda_2 = \mu_1\mu_2$. *Then the groups* $K(\lambda_1, \lambda_2)$, $K(\mu_1, \mu_2)$ *are conjugate.*

Proof. Set $\alpha = \sqrt{(\mu_1/\lambda_1)} = \sqrt{(\lambda_2/\mu_2)}$, $D = \left[\begin{smallmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{smallmatrix}\right]$. Then a brief calculation shows that

$$DS(\lambda_1)D^{-1} = S(\mu_1), \qquad DW(\lambda_2)D^{-1} = W(\mu_2),$$

so that $DK(\lambda_1, \lambda_2)D^{-1} = K(\mu_1, \mu_2)$. This completes the proof.

Thus by choosing $\mu_1 = \mu_2 = \sqrt{(\lambda_1\lambda_2)}$, we find that $K(\lambda_1, \lambda_2)$ and $K(\lambda)$, $\lambda = \sqrt{(\lambda_1\lambda_2)}$, are conjugate.

Our ultimate objective will be to prove the following:

Theorem 1. *The group* $K(\lambda_1, \lambda_2)$ *is discrete if and only if* $\lambda_1\lambda_2 \geq 4$, *or* $\lambda_1\lambda_2 = 4\cos^2(\pi/p)$, *where* p *is an integer* ≥ 3. $K(\lambda_1, \lambda_2)$ *is then structured as follows*:

Case 1. $\lambda_1\lambda_2 \geq 4$. *Then* $K(\lambda_1, \lambda_2)$ *is conjugate to a subgroup of index* 2 *of the Hecke group* $H(\lambda)$, $\lambda = \sqrt{(\lambda_1\lambda_2)}$.

Case 2. $\lambda_1\lambda_2 = 4\cos^2(\pi/p)$, p *even. Then* $K(\lambda_1, \lambda_2)$ *is conjugate to a subgroup of index* 2 *of the Hecke group* $H(\lambda)$, $\lambda = 2\cos(\pi/p)$.

Case 3. $\lambda_1\lambda_2 = 4\cos^2(\pi/p)$, p *odd. Then* $K(\lambda_1, \lambda_2)$ *is conjugate to the Hecke group* $H(2\cos(\pi/p))$ *and so is the free product of a cyclic group of order* 2 *and a cyclic group of order* p.

We first prove some purely group-theoretic results.

Theorem 2. *Let* $H = \{x, y\}$ *be a group generated by elements* x, y, *where* $x^2 = 1$. *Let* $K = \{xyx, y\}$. *Then* K *is a normal subgroup of* H, $(H : K) = 1$ *or 2, and* $H = K$ *if and only if* x *belongs to* K. *If* x *does not belong to* K, *then* $H = K + xK$.

Proof. We have

$$xKx^{-1} = \{y, xyx\} = K, \quad \text{since } x^2 = 1,$$

and

$$yKy^{-1} = K, \quad \text{since } y \text{ belongs to } K.$$

Hence K is a normal subgroup of H. Thus H/K is the group generated by x, y with the relations $x^2 = 1$, $xyx = 1$, $y = 1$; and these imply that H/K is of order 1 if x belongs to K, 2 otherwise. This completes the proof.

We can determine exactly when $H = K$, if H is a free product.

Theorem 3. *Put $u = xy$, and suppose in addition that $H = \{x\} * \{u\}$, the free product of the cyclic group $\{x\}$ of order 2 and the cyclic group $\{u\}$. Then $H = K$ if $\{u\}$ is finite and of odd order and $(H : K) = 2$, $H = K + xK$, if $\{u\}$ is finite and of even order, or if $\{u\}$ is infinite.*

Proof. Note first that $H = \{x, u\}$ and that $K = \{ux, xu\} = \{xu, u^2\}$.

Suppose first that $\{u\}$ is of order p, where p is odd. Then u^2 belongs to K, u^p belongs to K, so u belongs to K. This in turn implies that x belongs to K, so $K = H$.

Next, suppose that $\{u\}$ is of order p, where p is even. Then if x belongs to K, there are integers $a_1, b_1, a_2, b_2, \ldots$ such that

$$x = (xu)^{a_1}(u^2)^{b_1}(xu)^{a_2}(u^2)^{b_2}\cdots. \tag{2}$$

The exponent sums of x modulo 2 and u modulo p on each side of this equation must agree, since $H = \{x\} * \{u\}$; so

$$\begin{aligned} &a_1 + a_2 + \cdots \text{ is odd},\\ &a_1 + a_2 + \cdots + 2b_1 + 2b_2 + \cdots \equiv 0 \mod p. \end{aligned}$$

This is a contradiction, however, since p is even, and so x does not belong to K. Thus (by Theorem 2), $(H : K) = 2$, $H = K + xK$.

Finally, suppose that $\{u\}$ is of infinite order so that $x^2 = 1$ is the only relation. Then as before, if x belongs to K, equation (2) implies that

$$\begin{aligned} &a_1 + a_2 + \cdots \text{ is odd},\\ &a_1 + a_2 + \cdots + 2b_1 + 2b_2 + \cdots = 0. \end{aligned}$$

But this is also a contradiction, and so x does not belong to K. Thus $(H : K) = 2$, $H = K + xK$. This completes the proof.

Proof of Theorem 1. We set up the correspondence

$$x \to T, \qquad y \to S(\lambda)$$

so that

$$xyx \to W(\lambda)^{-1}, \quad H \to H(\lambda), \quad K \to K(\lambda).$$

It is classical that the Hecke group $H(\lambda)$ is discrete if and only if $\lambda \geq 2$ or $\lambda = 2\cos(\pi/p)$, where p is an integer ≥ 3. It is also clear that if F, G are arbitrary subgroups of $\mathrm{PSL}(2, R)$ such that F is discrete and $G \subset F$, then G is also discrete. These remarks, together with Lemma 1 and Theorems 2 and 3, constitute the proof of Theorem 1.

Some final remarks:

(a) It is known that $K(\lambda_1, \lambda_2)$ is free when $\lambda_1\lambda_2 \geq 4$.

(b) $K(\lambda_1, \lambda_2)$ and the Hecke group $H(\lambda)$, $\lambda = \sqrt{(\lambda_1\lambda_2)}$, are of the same "kind".

(c) $K(\lambda_1, \lambda_2)$ is always the free product of two cyclic groups.

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