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SUBSPACES AND GRAPHS

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ABSTRACT. Subspaces sufficiently near an arbitrary (fixed) subspace of a Hilbert space are shown to be in one-to-one correspondence with operators defined on the given subspace. Specifically, the nearby subspaces can be regarded as the graphs of these operators. This is applied to explicitly define a C^∞-atlas of charts for the set of subspaces.

1. INTRODUCTION

Given (real or complex) Hilbert spaces H and K, let $\mathscr{L}(H, K)$ denote the Banach space of (bounded linear) operators $T : H \to K$; we write $\mathscr{L}(H)$ for $\mathscr{L}(H, H)$. If $T \in \mathscr{L}(H, K)$, let $G(T)$ be the graph of T; thus

$$G(T) = \{(x, Tx) \in H \oplus K : x \in H\}.$$

Throughout, H is assumed to be a Hilbert space and we denote inner products by $\langle \cdot, \cdot \rangle$. (Orthogonal) projection onto a (closed linear) subspace M is denoted P_M. By identifying subspaces with projections, we can induce various topologies on the set $\mathscr{C}(H)$ of subspaces of H. Let us metrise $\mathscr{C}(H)$ by referring to the norm topology on $\mathscr{L}(H)$.

The present work concerns the relation of subspaces to a fixed subspace M by taking graphs of operators in $\mathscr{L}(M, M^\perp)$. The identification of H with the Hilbert direct sum $M \oplus M^\perp$ allows us to view such graphs as subspaces of H. This problem is studied by Halmos in [2], where only subspaces in "generic position" are considered. We consider the problem more generally, showing that a subspace N is the graph of an operator in $\mathscr{L}(M, M^\perp)$ if and only if $\|P_N - P_M\| < 1$.

Let $T \in \mathscr{L}(M, M^\perp)$. Then

$$G(T) = \{Sx : x \in M\} = SM$$

where S is the invertible operator in $\mathscr{L}(H)$ defined by $Sx = x + TP_M x$. The theory developed by Longstaff [4, 5] is applicable here. For our purposes, it is more convenient to think of $G(T)$ as the range of the operator $1 + T$ in $\mathscr{L}(M, H)$ defined by $(1 + T)x = x + Tx$, so we shall write $(1 + T)M$ for $G(T)$.

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2. The main result

Before proceeding to prove our main result, we need a preparatory lemma.

Lemma 1. *Let M be a subspace of H and let $T \in \mathscr{L}(M, M^\perp)$. Put $N = (1+T)M$. Then*

$$\|P_N - P_M\|^2 \le \frac{\|T\|^2}{1+\|T\|^2}.$$

Proof. It is easily verified that relative to the Hilbert direct sum $H = M \oplus M^\perp$,

$$N = \{(x, Tx) : x \in M\}, \qquad N^\perp = \{(-T^*y, y) : y \in M^\perp\}.$$

Let $(x, y) \in M \oplus M^\perp$. Then $P_N(x, y) = (u, Tu)$ and $P_{N^\perp}(x, y) = (-T^*v, v)$ for suitable $u \in M$ and $v \in M^\perp$. Since $P_N + P_{N^\perp}$ is the identity operator, it follows that $u - T^*v = x$ and $Tu + v = y$. We have

$$\begin{aligned}\|(x, y)\|^2 &= \|P_N(x, y)\|^2 + \|P_{N^\perp}(x, y)\|^2 = \|(u, Tu)\|^2 + \|(-T^*v, v)\|^2 \\ &= \|u\|^2 + \|Tu\|^2 + \|T^*v\|^2 + \|v\|^2 \le (1+\|T\|^2)(\|u\|^2 + \|v\|^2).\end{aligned}$$

Now $(P_N - P_M)(x, y) = (u - x, Tu) = (T^*v, Tu)$, so if $(x, y) \ne 0$, then

$$\|(P_N - P_M)(x, y)\|^2 = \|T^*v\|^2 + \|Tu\|^2 = \|(x, y)\|^2 - \|u\|^2 - \|v\|^2,$$

whence

$$\frac{\|(P_N - P_M)(x, y)\|^2}{\|(x, y)\|^2} = 1 - \frac{\|u\|^2 + \|v\|^2}{\|(x, y)\|^2} \le 1 - \frac{1}{1+\|T\|^2} = \frac{\|T\|^2}{1+\|T\|^2}. \quad \square$$

Solving for u in the above proof leads to the following matrix representation for P_N relative to the decomposition $H = M \oplus M^\perp$:

$$\begin{bmatrix} (1+T^*T)^{-1} & (1+T^*T)^{-1}T^* \\ T(1+T^*T)^{-1} & T(1+T^*T)^{-1}T^* \end{bmatrix}.$$

Note the similarity of this matrix representation to the one given by Halmos in [2, p. 386]. We now classify all subspaces that are graphs of operators in $\mathscr{L}(M, M^\perp)$.

Theorem 1. *Let M and N be subspaces of H. The following are equivalent:*

(1) $\|P_N - P_M\| < 1$.
(2) *N and $M^\perp$ are complementary ($N + M^\perp = H$ and $N \cap M^\perp = \{0\}$).*
(3) *$P_M|_N$ is injective and $P_M N = M$.*
(4) *$N = (1+T)M$ for some $T \in \mathscr{L}(M, M^\perp)$; T is unique and satisfies*

$$\|T\| = \frac{\|P_N - P_M\|}{\sqrt{1 - \|P_N - P_M\|^2}}.$$

Proof. (1) implies (2): Suppose $\|P_N - P_M\| < 1$ and let $x \in M^\perp \cap N$; if $x \ne 0$, then $\|P_N - P_M\| \ge \|(P_N - P_M)x\|/\|x\| = \|x\|/\|x\| = 1$, a contradiction. Thus $N \cap M^\perp = \{0\}$.

Now $M \ominus P_M N \subseteq N \cap M^\perp$ (if $x \in M \ominus P_M N$ then $\langle x, y\rangle = \langle P_M x, y\rangle = \langle x, P_M y\rangle = 0$ for all $y \in N$), so $P_M N$ is dense in M.

If $x \in N$, then $(1 - P_M)x = (P_N - P_M)x$, so

$$\|x\|^2 = \|P_M x\|^2 + \|P_{M^\perp} x\|^2 \le \|P_M x\|^2 + \|P_N - P_M\|^2 \|x\|^2,$$

whence $\|P_M x\| \ge \|x\| \sqrt{1 - \|P_N - P_M\|^2}$. Given a Cauchy sequence $\{P_M x_n\}_{n=1}^\infty$ in $P_M N$ with the $x_n \in N$, we see that $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence in N. Thus $x_n \to x$ for some $x \in N$, and then $P_M x_n \to P_M x \in P_M N$, so $P_M N$ is complete and therefore closed. Since $P_M N$ is dense in M, $P_M N = M$.

If $x \in H$, then $P_M x = P_M y$ for some $y \in N$ by the preceding result. Hence $x = P_M x + P_{M^\perp} x = y + P_{M^\perp}(x - y) \in N + M^\perp$, so $N + M^\perp = H$.

(2) implies (3): Suppose $x \in \ker P_M|_N$. Then $x \in N$ and $x \in M^\perp$, whence $x = 0$. Therefore $P_M|_N$ is injective. Now let $x \in M$. Then $x = y + z$ for suitable $y \in N$ and $z \in M^\perp$. We have $x = P_M x = P_M y \in P_M N$, so $P_M N = M$.

(3) implies (4): Assuming (3), the Banach Inversion Theorem asserts that $P_M|_N$ is invertible as an operator in $\mathscr{L}(N, M)$. Put $T = P_{M^\perp}(P_M|_N)^{-1} \in \mathscr{L}(M, M^\perp)$. Then

$$(1 + T)M = (P_M(P_M|_N)^{-1} + P_{M^\perp}(P_M|_N)^{-1})M = 1N = N.$$

It is clear that T is unique since graphs of different operators in $\mathscr{L}(M, M^\perp)$ are different. To calculate the norm of T, we use the fact that if E is a subset of $\mathbf{R}$ and f is a continuous increasing real-valued function defined on (the closure) $\overline{E}$, then $\sup f(E) = f(\sup E)$. For our application, $f(\xi) = \xi/\sqrt{1 - \xi^2}$ and $E = \{\|(P_N - P_M)y\|/\|y\| : y \in H \setminus \{0\}\}$. Since $\sup E = \|P_N - P_M\|$,

$$\begin{aligned}
\|T\| &= \sup\left\{\frac{\|P_{M^\perp}(P_M|_N)^{-1}x\|}{\|x\|} : x \in M \setminus \{0\}\right\} \\
&= \sup\left\{\frac{\|P_{M^\perp} y\|}{\|P_M y\|} : y \in N \setminus \{0\}\right\} \\
&= \sup\left\{\frac{\|(P_N - P_M)y\|/\|y\|}{\sqrt{1 - (\|(P_N - P_M)y\|/\|y\|)^2}} : y \in N \setminus \{0\}\right\} \\
&\le \sup\left\{\frac{\|(P_N - P_M)y\|/\|y\|}{\sqrt{1 - (\|(P_N - P_M)y\|/\|y\|)^2}} : y \in H \setminus \{0\}\right\} \\
&= \frac{\|P_N - P_M\|}{\sqrt{1 - \|P_N - P_M\|^2}}.
\end{aligned}$$

On the other hand, Lemma 1(2) yields the reverse inequality, so (4) follows.

(4) implies (1): By Lemma 1(2),

$$\|P_N - P_M\| \le \frac{\|T\|}{\sqrt{1 + \|T\|^2}} < 1. \quad \square$$

Corollary 1. *Let M and N be subspaces of H such that $\|P_N - P_M\| < 1$. Suppose S is an operator in $\mathscr{L}(M, M^\perp)$ satisfying $(1 + S)M \subseteq N$. Then $(1 + S)M = N$.*

Proof. Let $T \in \mathscr{L}(M, M^\perp)$ satisfy $(1 + T)M = N$; T exists by Theorem 1. Hence $G(S) \subseteq G(T)$, but this holds if and only if $S = T$. $\square$

The condition of Theorem 1(1) should be viewed in terms of the following

Proposition 1. *For any subspaces M and N of H, $\|P_N - P_M\| \leq 1$.*

Proof. Let $U_M = 2P_M - 1$ and $U_N = 2P_N - 1$. It is easily verified that U_M and U_N are unitary operators (indeed, they are symmetries). Now $P_N - P_M = \frac{1}{2}(U_N - U_M)$, so $\|P_N - P_M\| = \frac{1}{2}\|U_N - U_M\| \leq \frac{1}{2}(\|U_N\| + \|U_M\|) = 1$. □

3. An application

We conclude by defining the structure of a C^∞-manifold on $\mathscr{C}(H)$ that is compatible with the metric topology. Given a subspace M, Theorem 1 provides a homeomorphism between subspaces near M and operators in the Banach space $\mathscr{L}(M, M^\perp)$; this will be our chart map. We shall need a preliminary

Lemma 2. *Let M be a subspace of H and let $T_0 \in \mathscr{L}(M, M^\perp)$. Put $N = (1 + T_0)M$. Then there is a neighbourhood W of T_0 such that for each $T \in W$ there exists an operator S in $\mathscr{L}(N, N^\perp)$ satisfying $(1 + T)M = (1 + S)N$. Specifically, if $\|T - T_0\| < \|T_0\|^{-1}$ and $\|P_{(1+T)M} - P_N\| < 1$, then*

$$S = (1 - T_0^*)(1 + TT_0^*)^{-1}(T - T_0)P_M|_N.$$

Proof. First put

$$W = \{T \in \mathscr{L}(M, M^\perp) : \|T - T_0\| < \|T_0\|^{-1} \text{ and } \|P_{(1+T)M} - P_N\| < 1\}.$$

The matrix representation following Lemma 1 shows that the map $T \mapsto P_{(1+T)M}$ is continuous, whence W is open.

Now $T_0T_0^*$ is a positive operator in $\mathscr{L}(M^\perp)$, so $1 + T_0T_0^*$ is invertible. The functional calculus for positive selfadjoint operators yields $\|(1 + T_0T_0^*)^{-1}\|^{-1} \geq 1$.

Suppose $T \in W$. Then $1 + TT_0^* \in \mathscr{L}(M^\perp)$ and

$$\|(1 + TT_0^*) - (1 + T_0T_0^*)\| \leq \|T - T_0\|\|T_0\| < 1 \leq \|(1 + T_0T_0^*)^{-1}\|^{-1}.$$

By a standard result (see, e.g., [1, pp. 584–585]), $1 + TT_0^*$ is invertible.

Consider the Hilbert direct sum $H = M \oplus M^\perp$, and put

$$S = (1 - T_0^*)(1 + TT_0^*)^{-1}(T - T_0)P_M|_N.$$

Let $(x, T_0x) \in N$. Then $S(x, T_0x) = (-T_0^*y, y)$ where $y = (1 + TT_0^*)^{-1} \times (T - T_0)x \in M^\perp$. Thus $(1 + TT_0^*)y = (T - T_0)x$ or, equivalently,

$$T_0x + y = T(x - T_0^*y).$$

Therefore,

$$(1 + S)(x, T_0x) = (x - T_0^*y, T_0x + y) = (z, Tz) \in (1 + T)M,$$

where $z = x - T_0^*y$. Hence $(1 + S)N \subseteq (1 + T)M$, so $(1 + S)N = (1 + T)M$ by Corollary 1. □

For each subspace M of H, let

$$V_M = \{K \in \mathscr{C}(H) : \|P_K - P_M\| < \tfrac{1}{4}\}.$$

By Theorem 1, $N \in V_M$ if and only if $N = (1+T)M$ for some $T \in W_M$ where

$$W_M = \left\{T \in \mathscr{L}(M, M^\perp) : \|T\| < \frac{1}{\sqrt{15}}\right\},$$

so we can define a map $\phi_M : V_M \to W_M$ by

$$\phi_M((1+T)M) = T.$$

Observe that ϕ_M is a homeomorphism.

The V_M cover $\mathscr{C}(H)$, and $\phi_M(V_M \cap V_N)$ is open for all subspaces M and N since $V_M \cap V_N$ is open. Suppose $V_M \cap V_N \neq \varnothing$. Then

$$\|P_N - P_M\| \le \|P_N - P_K\| + \|P_K - P_M\| < \tfrac{1}{2}$$

for any $K \in V_M \cap V_N$. Thus $N = (1+T_0)M$ for some $T_0 \in \mathscr{L}(M, M^\perp)$ satisfying $\|T_0\| < \frac{1}{\sqrt{3}}$. Let $K \in V_M \cap V_N$. Then $K = (1+T)M$ for some $T \in \mathscr{L}(M, M^\perp)$ with $\|T\| < \frac{1}{\sqrt{15}}$ and $\|P_{(1+T)M} - P_N\| = \|P_K - P_N\| < \frac{1}{4} < 1$. Moreover,

$$\|T - T_0\| < \frac{1}{\sqrt{15}} + \frac{1}{\sqrt{3}} < \sqrt{3} < \|T_0\|^{-1},$$

so $K = (1+T)M = (1+S)N$ by Lemma 2, where $S = (1 - T_0^*) \times (1 + TT_0^*)^{-1}(T - T_0)P_M|_N$. This holds for all $K \in V_M \cap V_N$ and hence for all $T \in \phi_M(V_M \cap V_N)$. Therefore, $\phi_N\phi_M^{-1} : \phi_M(V_M \cap V_N) \to \phi_N(V_M \cap V_N)$ is given by

$$\phi_N\phi_M^{-1}(T) = (1 - T_0^*)(1 + TT_0^*)^{-1}(T - T_0)P_M|_N,$$

which is a C^∞ function. We have just proved

Theorem 2. *The metric space $\mathscr{C}(H)$ is a C^∞-manifold with the structure defined by the atlas consisting of the charts (V_M, ϕ_M) for each $M \in \mathscr{C}(H)$.*

In fact, $\mathscr{C}(H)$ is a complex analytic manifold, at least in the case where H is a finite-dimensional complex Hilbert space. The above C^∞-atlas is fundamental to Noakes's definition of stability of invariant subspaces in [6]: if $A \in \mathscr{L}(H)$ and M is an invariant subspace of A, then the pair (M, A) is *stable* if there is a C^1 function $f : V \to \mathscr{L}(M, M^\perp)$ defined on an (open) neighbourhood V of A such that $f(A) = 0$ and $(1 + f(B))M$ is B-invariant for each $B \in V$. The definition of stability simplifies to the requirement that a C^1 function $g : V \to \mathscr{C}(H)$ exists such that $g(A) = M$ and $g(B)$ is B-invariant for all $B \in V$; in such a case, it is evident that the pair $(g(B), B)$ is stable for all $B \in V$.

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