PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 119, Number 1, September 1993

SOME ESTIMATES FOR HARMONIC MEASURES. III

JAMES A. JENKINS

(Communicated by Clifford J. Earle, Jr.)

ABSTRACT. A simple determination is given for continua in the closed unit disc having minimal harmonic measure at the centre.

1

Dieter Gaier [2] raised the following problem. Let a point on the unit circumference and a point in the unit disc distinct from the origin be given. Find a continuum containing these points whose harmonic measure at the origin is minimal. This problem was solved by the author [3], who showed that extremal continua are made up of trajectories of appropriate quadratic differentials and that apart from certain special cases there is a unique extremal configuration. A more general problem also raised by Gaier was treated by Liao [5]. A special case of a problem of Fuchs [1] is to find a continuum in the closed unit disc which meets every radius and whose harmonic measure at the origin is minimal. The author [4] characterized the class of continua in which such an extremal continuum must occur. Marshall and Sundberg [6] determined the essentially unique extremal continuum. In this paper we give a much simplified derivation of this solution using the method of the extremal metric.

2

We follow the normalization and notation used in [4, Theorem 2]. In particular, we consider a potential extremal continuum Γ with end points at 1 and r, $0 < r < 1$, consisting of the arc on the unit circumference made up of points $e^{i\theta}$ with $0 \leq \theta \leq \chi$, $\pi \leq \chi < 2\pi$, and an arc γ on a trajectory of the appropriate quadratic differential from $e^{i\chi}$ to r. The minimal harmonic measure is equivalent to a maximal triad module $m(0, \alpha, D)$ where α is the open arc made up of points $e^{i\theta}$ with $\chi < \theta < 2\pi$ and D is the unit disc slit along γ.

We consider the quadratic differential

$$Q(z)\,dz^2 = \frac{i(z+1)}{z(z-1)(z-r)(z-r^{-1})}\,dz^2$$

Received by the editors January 28, 1992.
1991 *Mathematics Subject Classification*. Primary 31A15, 30C85.

and the comparison metric $|Q(z)|^{1/2}|dz|$ in the above module problem. We pass to a two-sheeted covering of the disc branched at 0 and r and obtain a quadrangle $\mathcal{Q}$ for which one pair of opposite sides is given by the two open arcs covering α. Its module for curves joining this pair of opposite sides is twice the above triad module. Let the segment $[-1, 0]$ have length a and the segment $[0, r]$ have length b in the Q-metric. Considering the mapping $\zeta = \int (Q(z))^{1/2}\,dz$ of the upper semicircle we see this has area $ab + \frac{1}{2}b^2$ in the Q-metric. Thus $\mathcal{Q}$ has area $4(ab + \frac{1}{2}b^2)$. Also we see that the segment $[r, 1]$ has length $a+b$ so that every curve joining the above sides in $\mathcal{Q}$ has length at least $2^{1/2}(a+2b)$. The module of $\mathcal{Q}$ is at most

$$4(ab + \tfrac{1}{2}b^2)/2(a+2b)^2 = (2ab^{-1}+1)(ab^{-1}+2)^{-2}.$$

An elementary calculation shows this quantity to have maximal value $\frac{1}{3}$, which can occur only if $a = b$ and further only if γ is a trajectory of $Q(z)\,dz^2$ and if this runs from -1 to r. It is seen at once that this situation can occur for at most one value of r since if there were two, r, r', we could map the upper semicircle onto itself keeping -1, 0, 1 fixed but carrying r to r'.

On the other hand, if we take the quadratic differential $w(w^6-1)^{-1}\,dw^2$, which has simple zeros at 0, ∞ and simple poles at the sixth roots of unity, it has the real axis to the right of 1 and the rays obtained by rotating this about the origin through $120°$ and $240°$ as trajectories. The same is true of the radius from 0 to -1 and the rotated radii. The transformation $\zeta = (w-i)(1-iw)^{-1}$ carries this to the quadratic differential $2i(\zeta^2+1)(\zeta^6 - 15\zeta^4 + 15\zeta^2 - 1)^{-1}\,d\zeta^2$ with the first set of trajectories going to the quarter unit circle from 1 to i and arcs running from i to points on the negative real axis. The transformation $z = \zeta^2$ transforms this to the quadratic differential

$$\tfrac{1}{2}i(z+1)[z(z-1)(z-r)(z-r^{-1})]^{-1}\,dz^2$$

where $r = 7 - 4\sqrt{3}$ and the trajectories go to the upper unit semicircle and arcs running from -1 to r inside the unit circle and to r^{-1} outside. (That the trajectory from -1 to r has the desired property is immediate by invariance, but one also verifies at once that $z = r$ is the image of $w = e^{2\pi i/3}$.) Thus this configuration provides the unique extremal in our problem.

Theorem. *Let F be the closed unit disc $|z| \le 1$, E the open unit disc, and C a continuum in F not containing the origin which meets every radius of F. Let G be the component of $E - C$ containing the origin and α the border entity of G determined by C. Then the harmonic measure $\omega(0, \alpha, G)$ attains its minimal value for the continuum $\tilde{C}$ consisting of the upper semicircle and the trajectory (with its end points) running from -1 to r of the quadratic differential*

$$i(z+1)[z(z-1)(z-r)(z-r^{-1})]^{-1}\,dz^2$$

with $r = 7 - 4\sqrt{3}$. The only other continua for which the minimum is attained are those derived from $\tilde{C}$ by rotation about the origin and reflection in lines through the origin.

3

The extreme value of the triad module is of course $1/6$. Marshall and Sundberg calculated the corresponding value of the harmonic measure to a great many decimal places. However, this quantity can be represented explicitly using the relationship between triad module and harmonic measure.

If the harmonic measure is $\alpha/2\pi$, we take the arc on the unit circumference from $e^{-i\alpha/2}$ to $e^{i\alpha/2}$ (in the positive sense) and the corresponding triad module. As before this is one half the module of the quadrangle with vertices $e^{-i\alpha/4}$, $e^{i\alpha/4}$, $-e^{-i\alpha/4}$, $-e^{i\alpha/4}$ for curves joining the sides from $e^{i\alpha/4}$ to $-e^{-i\alpha/4}$ and from $-e^{i\alpha/4}$ to $e^{-i\alpha/4}$ on the unit circumference. We map this onto the upper half-plane by the linear transformation $w = i\cot\frac{\alpha}{8}\frac{1-z}{1+z}$ so that the vertices go to 1, k^{-1}, $-k^{-1}$, -1, with $k = \cot^2\frac{\alpha}{8}$. The elliptic integral

$$\zeta = \int_0 [(1-w^2)(1-k^2w^2)]^{-1/2}\,dw$$

maps this on the rectangle

$$-K < \mathscr{R}\zeta < K, \qquad 0 < \mathscr{I}\zeta < K',$$

where K and K' are complete elliptic integrals corresponding to $k = \cot^2\frac{\alpha}{8}$. The module of this corresponding to the above is $\frac{K'}{2K}$. For this to be $\frac{1}{3}$ we have $\frac{K'}{K} = \frac{2}{3}$.

BIBLIOGRAPHY

1. K. Barth, D. Brannan, and W. Hayman, *Research problems in complex analysis*, Bull. London Math. Soc. **16** (1984), 490–517.
2. D. Gaier, *Estimates of conformal mappings near the boundary*, Indiana Univ. Math. J. **21** (1972), 581–595.
3. James A. Jenkins, *On a problem concerning harmonic measure*, Math. Z. **135** (1974), 279–283.
4. ——, *Some estimates for harmonic measures*, Complex Analysis 1, Proceedings of the special year held at the University of Maryland (College Park, 1985–86), Lecture Notes in Math., vol. 1275, Springer-Verlag, New York, 1987, pp. 210–214.
5. Luby Liao, *Certain extremal problems concerning module and harmonic measure*, J. Analyse Math. **40** (1981), 1–42.
6. D. Marshall and C. Sundberg, *Harmonic measure and radial projection*, Trans. Amer. Math. Soc. **316** (1989), 81–95.

DEPARTMENT OF MATHEMATICS, WASHINGTON UNIVERSITY, ST. LOUIS, MISSOURI 63130