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VANISHING THEOREMS FOR SINGULAR VARIETIES

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ABSTRACT. We give generalizations of Ramanujam's version of the Kodaira Vanishing Theorem to singular varieties.

In this note, we give generalizations of the Kodaira Vanishing Theorem to singular varieties. Let X be an n-dimensional Cohen-Macaulay projective variety over $\mathbb{C}$ and L be a line bundle on X. For $s \in V = H^0(X, L^m)$, $s \neq 0$, let D_s be the zero-scheme of s, a Cartier divisor on X. Let $B = \bigcap_{s \in V - \{0\}} D_s$ be the base locus of the linear system V. Then, there is a rational map $\phi_{L^m} : X \to \mathbb{P}(V^*)$ defined outside B by sending x to the hyperplane of divisors containing x.

Theorem 1. *Let X, L, and ϕ_{L^m} be as above and B be the base locus of the linear system $H^0(X, L^m)$. Suppose that the dimension of the image of X under the map ϕ_{L^m} is at least k. Then*

$$H^i(X, \omega_X \otimes L) = 0 \quad \textit{for } i > \max(n-k, \dim B, \dim \operatorname{Sing}(X)),$$

where $\operatorname{Sing}(X)$ *denotes the singular locus of* X.

Remark 1. When X is a nonsingular projective variety over $\mathbb{C}$, this was proved by Ramanujam [R, Theorem 3].

Proof. First of all, we reduce the case to $m = 1$ using a finite covering of X. By Bertini's Theorem, there is a basis $\{s_0, \dots, s_r\}$ of the complete linear system $H^0(X, L^m)$ such that the zero-scheme D_{s_i} of s_i is smooth outside the union of B and $\operatorname{Sing}(X)$ for $i = 0, \dots, r$. Now we construct a tower of cyclic coverings inductively. We set $Y_{-1} = X$ and τ_{-1} to be the identity map on X. The section $\tau_{j-1}^*(s_j)$ of $\tau_{j-1}^*(L^m)$ defines a structure of an $\mathcal{O}_{Y_{j-1}}$-algebra on $\bigoplus_{i=0}^{m-1} \tau_{j-1}^* L^{-i}$. The natural map

$$\sigma_j : Y_j = \operatorname{Spec}_{Y_{j-1}} \left(\bigoplus_{i=0}^{m-1} \tau_{j-1}^* L^{-i} \right) \to Y_{j-1}$$

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gives an m-fold cyclic covering map of Y_{j-1}. Let

$$\tau_j = \sigma_j \circ \tau_{j-1} : \ Y_j \to X$$

be the composition map and $\tau = \tau_r$. Then Y_r will be smooth outside the inverse image $\tau^{-1}(B \cup \mathrm{Sing}(X))$. In particular, the maximum of the dimensions of B and $\mathrm{Sing}(X)$ will be preserved under the map τ. Also note that $\tau^* L$ possesses sections $t_0, \dots, t_r$ such that $t_i^m = \tau^*(s_i)$ in $\tau^* L^m$. Thus the field $\mathbb{C}(t_1/t_0, \dots, t_r/t_0)$ is algebraic over the function field $\mathbb{C}(s_1/s_0, \dots, s_r/s_0)$ of $\phi_{L^m}(X)$. Hence the dimension of the image of Y_r under the map $\phi_{\tau^* L}$ is at least k since the function field $K(\phi_{\tau^* L}(Y_r))$ contains $\mathbb{C}(t_1/t_0, \dots, t_r/t_0)$. Furthermore, we have

$$H^i(X, \tau_* \tau^* L^{-1}) = H^i(Y_r, \tau^* L^{-1}),$$

since τ is affine. The trace map gives a splitting of the natural homomorphism $L^{-1} \to \tau_* \tau^* L^{-1}$, and hence L^{-1} is a direct summand of $\tau_* \tau^* L^{-1}$. Moreover, Y_r is Cohen-Macaulay since τ is flat. Now by Serre duality, $H^i(Y_r, \omega_{Y_r} \otimes \tau^* L) = 0$ will imply that $H^i(X, \omega_X \otimes L) = 0$. Hence we may assume $m = 1$.

Let B be the base locus of the linear system $V = H^0(X, L)$. Let $\pi : \widetilde{X} \to X$ be a desingularization of the blow-up of X along B. There is a morphism

$$\psi : \ \widetilde{X} \to \mathbb{P}(V^*)$$

extending the morphism $\phi_L : \ X - B \to \mathbb{P}(V^*)$. Then we have

$$\psi^* \mathscr{O}(1) = \pi^* L \otimes \mathscr{O}(-E)$$

where E is an effective divisor on $\widetilde{X}$ with support in $\pi^{-1}(B)$. Thus there is an exact sequence

$$(1) \qquad 0 \to \psi^* \mathscr{O}(1) \otimes \omega_{\widetilde{X}} \to \pi^* L \otimes \omega_{\widetilde{X}} \to \pi^* L \otimes \omega_{\widetilde{X}} \otimes \mathscr{O}_E \to 0.$$

We claim that

$$R^q \pi_* (\psi^* \mathscr{O}(1) \otimes \omega_{\widetilde{X}}) = 0 \quad \text{for } q > 0.$$

Let H be a smooth divisor of a section of $\psi^* \mathscr{O}(1)$. Consider the Poincaré residue sequence

$$0 \to \omega_{\widetilde{X}} \to \mathscr{O}(H) \otimes \omega_{\widetilde{X}} \to \omega_H \to 0.$$

Grauert-Riemenschneider Vanishing Theorem [GR, Satz 2.3] implies that

$$R^q \pi_* \omega_{\widetilde{X}} = R^q \pi_* \omega_H = 0 \quad \text{for } q > 0.$$

Thus we conclude that

$$R^q \pi_* (\psi^* \mathscr{O}(1) \otimes \omega_{\widetilde{X}}) = R^q \pi_* (\mathscr{O}(H) \otimes \omega_{\widetilde{X}}) = 0 \quad \text{for } q > 0.$$

Hence we get from (1)

$$0 \to \pi_* (\psi^* \mathscr{O}(1) \otimes \omega_{\widetilde{X}}) \to \pi_* (\pi^* L \otimes \omega_{\widetilde{X}}) \to \pi_* (\pi^* L \otimes \omega_{\widetilde{X}} \otimes \mathscr{O}_E) \to 0.$$

Since $\pi_* (\pi^* L \otimes \omega_{\widetilde{X}} \otimes \mathscr{O}_E)$ is supported on B, there is a surjection

$$H^i(X, \pi_* (\psi^* \mathscr{O}(1) \otimes \omega_{\widetilde{X}})) \to H^i(X, \pi_* (\pi^* L \otimes \omega_{\widetilde{X}})) \quad \text{for } i > \dim B.$$

(In fact, this is an isomorphism for $i > \dim B + 1$.) Via the Leray spectral sequence, this induces a surjection

$$H^i(\widetilde{X}, \psi^*\mathscr{O}(1) \otimes \omega_{\widetilde{X}}) \to H^i(\widetilde{X}, \pi^*L \otimes \omega_{\widetilde{X}}) \quad \text{for } i > \dim B, \tag{2}$$

since $R^q\pi_*(\pi^*L \otimes \omega_{\widetilde{X}}) = L \otimes R^q\pi_*\omega_{\widetilde{X}} = 0$, $q > 0$, by the Grauert-Riemenschneider Vanishing Theorem [GR, Satz 2.3].

On the other hand, we have

$$H^i(\widetilde{X}, \psi^*\mathscr{O}(1) \otimes \omega_{\widetilde{X}}) = 0 \quad \text{for } i > n - k, \tag{3}$$

by Ramanujam's generalization of the Kodaira Vanishing Theorem [R, Theorem 3]. Therefore, we have

$$H^i(\widetilde{X}, \pi^*L \otimes \omega_{\widetilde{X}}) = 0 \quad \text{for } i > \max(n - k, \dim B).$$

Again from the Leray spectral sequence and the Grauert-Riemenschneider Vanishing Theorem [GR, Satz 2.3], it follows that

$$H^i(X, L \otimes \pi_*\omega_{\widetilde{X}}) = 0 \quad \text{for } i > \max(n - k, \dim B). \tag{4}$$

Now our proof follows from an exact sequence

$$0 \to \pi_*\omega_{\widetilde{X}} \to \omega_X \to Q \to 0$$

where Q is supported on $\operatorname{Sing}(X)$. □

Corollary 2. *Let X be an n-dimensional projective variety over $\mathbb{C}$ with only rational singularities outside the base locus B of $H^0(X, L^m)$. Suppose that the dimension of the image of X under the map ϕ_{L^m} is at least k. Then*

$$H^i(X, \omega_X \otimes L) = 0 \quad \textit{for } i > \max(n - k, \dim B).$$

Proof. First, we will show that the reduction step to $m = 1$ works. Let $\pi : \widetilde{X} \to X$ be a desingularization of the blow-up of X along B. By Bertini's Theorem, there is a basis $\{s_0, \dots, s_r\}$ of $H^0(X, L^m)$ such that the zero schemes of $\pi^*(s_i)$ are of the form $H_i + E$ where H_i are smooth hypersurfaces and E is supported on $\pi^{-1}(B)$. Let $\tau : Y_r \to X$ be a branched covering of X over the union of the divisors D_{s_i} constructed as in the proof of the theorem. Consider the fibre product:

$$\begin{array}{ccc} \widetilde{Y}_r & \xrightarrow{\tilde{\tau}} & \widetilde{X} \\ \downarrow p & & \downarrow \pi \\ Y_r & \xrightarrow{\tau} & X \end{array}$$

Since cohomology commutes with flat base extension [H, III, 9.3], there are natural isomorphisms

$$R^ip_*\mathscr{O}_{\widetilde{Y}_r} \cong \tau^*R^i\pi_*\mathscr{O}_{\widetilde{X}}. \tag{5}$$

Since X has only rational singularities outside B, we have that $R^ip_*\mathscr{O}_{\widetilde{Y}_r}$ is supported on $\tau^{-1}(B)$ when $i > 0$ and $p_*\mathscr{O}_{\widetilde{Y}_r} = \mathscr{O}_{Y_r}$ on $Y_r - \tau^{-1}(B)$. Moreover, $\widetilde{Y}_r$ is smooth outside $\tilde{\tau}^{-1}(E)$ since the zero schemes of $\pi^*(s_i)$ are smooth on $X - E$. Thus Y_r has only rational singularities outside $\tau^{-1}(B)$. Note that the dimension of B is unchanged under the map τ. Hence we may assume $m = 1$.

From [K, Proposition, p. 50], we obtain an exact sequence

$$0 \to \pi_* \omega_{\tilde{X}} \to \omega_X \to Q \to 0 \tag{6}$$

where Q is supported on the base locus B. The corollary follows from (4) and this sequence. □

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