ON $\varphi$-EXTENSIONS OF DEVELOPABLE SPACES

T. MIZOKAMI

(Communicated by Franklin D. Tall)

Abstract. We prove that the $\varphi$-extension of Moore spaces is a developable space.

1. Introduction

In this paper, all the spaces are $T_1$ and all mappings are continuous and onto. The letter $N$ denotes the positive integers.

For a class $\mathcal{C}$ of topological spaces, Klebanov called a space $Y$ the $\varphi$-extension of $\mathcal{C}$ if $Y$ is the image of a product space of members of $\mathcal{C}$ under a closed mapping $[K_1]$. He proved that a $\varphi$-extension of metric spaces is metrizable if it is first countable $[K_1$, Theorem 6; $K_2]$. From this result, the following question arises: For what classes $\mathcal{C}$ does the $\varphi$-extension $Y$ of $\mathcal{C}$ belong to $\mathcal{C}$ if $Y$ is first countable? The author recently obtained positive results for classes of Lašnev spaces, stratifiable spaces, and paracompact $\sigma$-spaces but posed the question for the class of developable spaces $[M]$.

In this paper, the author gives a positive answer to the question when $\mathcal{C}$ is a class of Moore spaces, that is, regular developable spaces. A developable space is a space $X$ with a sequence $\{\mathbb{Z}_n : n \in N\}$ of open covers such that, for each point $p \in X$, $\{St(p, \mathbb{Z}_n) : n \in N\}$ is a local base at $p$ in $X$.

For the sake of brevity, we write $[X_\alpha, A, f, Y]$ for the situation that $Y$ is the image of the product space $\prod\{X_\alpha : \alpha \in A\}$ under a closed mapping $f$. For an index set $A$, the product space $\prod\{X_\alpha : \alpha \in A\}$ is briefly denoted by $X(A)$.

For a family $\mathcal{H}$ of subsets of a space $X$ and a point $p \in X$, we write

$$C(p, \mathcal{H}) = \bigcap\{H \in \mathcal{H} : p \in H\}.$$
Lemma 1. For a space $X$, the following are equivalent:

1. $X$ is a developable space.
2. There exists a $\sigma$-discrete pair network for $X$.
3. There exists a $\sigma$-locally finite pair network for $X$.

For Lemma 2, assume $[X_\alpha, A, f, Y]$. For each $\alpha \in A$ let $\mathcal{F}_\alpha$ be a locally finite family of nonempty closed subsets of $X_\alpha$, and for each $k \in \mathbb{N}$ let $\Delta_k$ be the totality of subsets $\delta$ of $A$ with $|\delta| = k$. For each $\delta \in \Delta_k$ let

$$Y_\delta = \{y \in Y : H \subset f^{-1}(y) \text{ for some } H \in \mathcal{H}(\delta)\},$$

where

$$\mathcal{H}(\delta) = \prod \{\mathcal{F}_\alpha : \alpha \in \delta\} \times \{X(A - \delta)\}.$$ 

Lemma 2. $\bigcup\{Y_\delta : \delta \in \Delta_k\}$ is a discrete closed subset of $Y$.

Proof. Without loss of generality, we can assume $Y_\delta \neq \emptyset$ for all $\delta \in \Delta_k$ and $Y_\delta \neq Y_{\delta'}$ if $\delta \neq \delta'$, $\delta, \delta' \in \Delta_k$. First we show that $\delta \cap \delta' \neq \emptyset$ for each $\delta, \delta' \in \Delta_k$. Take points $y \in Y_\delta$, $y' \in Y_{\delta'}$ such that $y \neq y'$. By the definition, there exist $H \in \mathcal{H}(\delta)$, $H' \in \mathcal{H}(\delta')$ such that

$$H \subset f^{-1}(y), \quad H' \subset f^{-1}(y'),$$

where

$$H = \prod \{F_\alpha : \alpha \in \delta\} \times X(A - \delta),$$

$$H' = \prod \{F'_\alpha : \alpha \in \delta'\} \times Y(A - \delta').$$

If $\delta \cap \delta' \neq \emptyset$ were true, then we would have

$$H \cap H' = \prod \{F_\alpha : \alpha \in \delta\} \times \prod \{F'_\alpha : \alpha \in \delta'\} \times X(A - (\delta \cup \delta')) \neq \emptyset,$$

which is a contradiction to $f^{-1}(y) \cap f^{-1}(y') = \emptyset$. Hence we have $\delta \cap \delta' \neq \emptyset$.

Since $|\delta| = k$ for each $\delta \in \Delta_k$, we can show inductively that, for some $t_k \in \mathbb{N}$, $\Delta_k$ can be written as $\Delta_k = \bigcup \{\Delta(i) : i \leq t_k\}$, where $\{\Delta(i)\}$ has the property that, for each $i \leq t_k$ there exists $\delta_i \subset A$ such that for each $\delta$, $\delta' \in \Delta(i)$ with $\delta \neq \delta'$ we have

$$\delta \cap \delta' = \delta_i \quad \text{and} \quad (\delta - \delta_i) \cap (\delta' - \delta_i) = \emptyset.$$

Since obviously each $Y_\delta$ is a discrete closed subset of $Y$, we can assume $Y_\delta \cap Y_{\delta'} = \emptyset$ if $\delta \neq \delta'$, $\delta, \delta' \in \Delta_k$. Let $i \leq t$ be fixed. For each $y \in Y_\delta$, $\delta \in \Delta(i)$, we take $H(y) \in \mathcal{H}(\delta)$ such that $H(y) \subset f^{-1}(y)$. $H(y)$ is of the form

$$H(y) = \prod \{F_\alpha(y) : \alpha \in \delta\} \times X(A - \delta),$$

where $F_\alpha(y) \in \mathcal{F}_\alpha$, $\alpha \in \delta$. If we set

$$T(y) = \prod \{F_\alpha(y) : \alpha \in \delta\} \times X(A - \delta_i),$$

then we easily have

$$Y_\delta \subset f \left( \bigcup \{T(y) : y \in Y_\delta\} \right).$$
From the property of $\Delta_i$ and the assumption that $Y_\delta \cap Y_{\delta'} = \emptyset$ for $\delta, \delta' \in \Delta_i, \delta \neq \delta'$, it follows that

$$\left\{ \bigcup \{T(y) : y \in Y_\delta\} : \delta \in \Delta_i \right\}$$

is hereditarily closure-preserving in $X(A)$. Therefore, $\{Y_\delta : \delta \in \Delta_i\}$ is closure-preserving in $Y$, and consequently

$$Y(\Delta_i) = \bigcup \{Y_\delta : \delta \in \Delta_i\}$$

is a discrete closed subset of $Y$. This means that

$$\bigcup \{Y_\delta : \delta \in \Delta_k\} = \bigcup \{Y(\Delta_i) : i \leq t\}$$

is a discrete closed subset of $Y$. This completes the proof.

We call a subset $Y$ of a space $X$ a $\sigma$-discrete closed subset of $X$ if $Y = \bigcup\{Y_n : n \in \mathbb{N}\}$, where each $Y_n$ is a discrete closed subset of $X$.

**Lemma 3.** In the situation $[X_\alpha, A, f, Y]$, we assume that all $X_\alpha$ are Moore spaces and $Y$ is first countable. Then $Y$ is decomposed as $Y = Y_0 \cup Y_1$, where $Y_1$ is a $\sigma$-discrete closed subset of $Y$ and, for each point $y \in Y_0$, $f^{-1}(y)$ is compact in $X(A)$.

**Proof.** For each $\alpha \in A$, since $X_\alpha$ is a $\sigma$-space in the sense of [O], $X_\alpha$ has a network $\mathcal{F}_\alpha = \bigcup\{\mathcal{F}_{\alpha n} : n \in \mathbb{N}\}$, where, for each $n$, $\mathcal{F}_{\alpha n}$ is a locally finite closed cover of $X_\alpha$ such that $\mathcal{F}_{\alpha n} \subseteq \mathcal{F}_{\alpha n+1}$ and any finite intersection of members of $\mathcal{F}_{\alpha n}$ belongs to $\mathcal{F}_{\alpha n}$. Let $\Delta$ be the totality of finite subsets of $A$, and for each $\delta \in \Delta$ with $|\delta| = k$ and each $n \in \mathbb{N}$ let

$$\mathcal{H}(\delta, n) = \prod\{\mathcal{F}_{\alpha n} : \alpha \in \delta\} \times (X(A - \delta))$$

and define a subset $Y(\delta, n)$ of $Y$ by

$$Y(\delta, n) = \{y \in Y : H \subset f^{-1}(y) \text{ for some } H \in \mathcal{H}(\delta, n)\}.$$

Set

$$Y(k, n) = \bigcup\{Y(\delta, n) : \delta \in \Delta_k\}, \quad k, n \in \mathbb{N},$$

$$Y_1 = \bigcup\{Y(k, n) : k, n \in \mathbb{N}\}, \quad Y_0 = Y - Y_1,$$

where $\Delta_k = \{\delta \in \Delta : |\delta| = k\}$. By Lemma 2, $Y_1$ is a $\sigma$-discrete closed subset of $Y$. Therefore, it remains to show that, for each $y \in Y_0$, $f^{-1}(y)$ is compact in $X(A)$. Assume that $y \in Y_0$ and $f^{-1}(y)$ is not compact in $X(A)$. Since a Moore space is subparacompact, there exists $\alpha_0 \in A$ such that

1. $p_0(f^{-1}(y))$ is not countably compact in $X_{\alpha_0}$,

where $p_0 : X(A) \to X_{\alpha_0}$ is the projection. We settle the following:

2. $p_0(f^{-1}(y))$ is Lindelöf.

Since $\mathcal{F}_{\alpha_0}$ is a network for $X_{\alpha_0}$, it suffices to show that for each $n$

$$\mathcal{F}_n(y) = \{F \in \mathcal{F}_{\alpha_0 n} : F \cap p_0(f^{-1}(y)) \neq \emptyset\}$$
is finite. Assume that, for some \( n \), \( \mathcal{F}_n(y) \) is infinite. Let \( \{O_n(y) : n \in N\} \) be a decreasing local base at \( y \) in \( Y \). We take \( \{F_m : m \in N\} \subset \mathcal{F}_n(y) \), which implies 

\[
p_0^{-1}(F_m) = H_m \in \mathcal{H}(\{\alpha_0\}, n) \quad \text{for each} \ m \in N.
\]

Observe that, for each \( m, n \in N \), \( f(H_m) \cap O_n(y) \) is infinite, for otherwise for some \( k, m \in N \)

\[
H_m \cap f^{-1}(O_k(y)) \subset f^{-1}(y),
\]

which implies that \( H \subset f^{-1}(y) \) for some \( H \in \mathcal{H}(\delta, \tau) \), that is, \( y \in Y_1 \), a contradiction. From this observation, we can take a sequence \( \{p_n : n \in N\} \subset X_\alpha \) such that \( p_n \in f^{-1}(O_n(y)) \cap H_n \) and \( f(p_n) \neq f(p_m) \) if \( n \neq m \). Then \( \{p_n\} \) clusters because of the closedness of \( f \). But this is a contradiction to the discreteness of \( \{H_m : m \in N\} \). Hence we can conclude (2). By (1), (2), and the regularity of \( X_\alpha \), we can easily find an increasing sequence \( \{U_k : k \in N\} \) of open subsets of \( X_\alpha \) covering \( p_0(f^{-1}(y)) \) and satisfying

\[
p_0(f^{-1}(y)) \cap (U_{k+1} - U_k) \neq \emptyset
\]

for each \( k \). Set \( V_k = p_0^{-1}(U_k), k \in N \). Since we can show that, for each \( k \), \( f^{-1}(O_k(y)) \cap (V_{k+1} - V_k) \) is infinite in the same way as above, we can take a sequence \( \{z_k : k \in N\} \subset X(A) \) such that

\[
z_k \in f^{-1}(O_k(y)) \cap (V_{k+1} - V_k), \quad k \in N.
\]

Since \( f \) is a closed mapping, \( \{z_k\} \) has a cluster point \( z_0 \) with \( z_0 \in f^{-1}(y) \). But this is a contradiction because \( z_0 \in V_k \) for some \( k \). This contradiction means that \( f^{-1}(y) \) is compact in \( X(A) \), completing the proof.

**The main theorem.** A first countable \( \varphi \)-extension of Moore spaces is a developable space.

**Proof.** In the situation \([X_\alpha, A, f, Y]\), we assume that all \( X_\alpha \) are Moore spaces and \( Y \) is a first countable space. By Lemma 3, we consider two cases as follows: (i) \( Y \) is a countable union of discrete closed subsets of \( Y \); and (ii) there exists a point \( y \in Y \) such that \( f^{-1}(y) \) is compact in \( X(A) \). In the first case, using the first countability of \( Y \) we can easily see that \( Y \) has a \( \sigma \)-discrete pair network. Therefore, by Lemma 1 \( Y \) is developable. Thus it remains to show that \( Y \) is developable under the case (ii). By [K1, Lemma 1], without loss of generality we can assume that \( Y \) is the image of the product space \( X \times X(A) \) under a closed mapping \( f \), where \( X \) is a Moore space and all \( X_\alpha \) are compact metrizable spaces. By Lemma 1, there exists a \( \sigma \)-locally finite pair network \( \{P_n : n \in N\} \) for \( X \) such that, for each \( n \), \( P_n = \{P_\lambda : \lambda \in \Lambda_n\} \) and \( (P_n)_1 = \{P_\lambda : \lambda \in \Lambda_n\} \) is a locally finite closed cover of \( X \). Without loss of generality, we can assume that \( \bigcup\{(P_n)_1 : n \in N\} \) is closed under any finite intersection. For each \( \lambda \in \Lambda = \bigcup\{\Lambda_n : n \in N\} \), we choose a point \( x_\lambda \in P_\lambda \) arbitrarily. Since

\[
Y(\lambda) = f(\{x_\lambda\} \times X(A))
\]

is a first countable dyadic space, by [N, Theorem VIII, 11] \( Y(\lambda) \) is a compact metrizable space. Let \( p_\lambda : \{x_\lambda\} \times X(A) \rightarrow X(A) \) be the projection, and let \( g_\lambda : X(A) \rightarrow f(\{x_\lambda\} \times X(A)) \subset Y \) be the mapping such that \( f/\{x_\lambda\} \times X(A)) = g_\lambda \cdot p_\lambda \). Since \( Y(\lambda) \) is a compact metrizable space, there exists a countable pair network \( \{Q_{\lambda n} = (Q_{\lambda n1}, Q_{\lambda n2}) : n \in N\} \) for \( Y(\lambda) \). Without loss of generality,
we can assume that, for each $n$, there exists a sequence $\{n_i : i \in N\} \subset N$ such that
\[
Q_{\lambda n_1} = Q_{\lambda n_2} = \cdots = Q_{\lambda n_1},
\]
(3) \quad Q_{\lambda n_1} \subset Q_{\lambda n_2}, \quad i \in N,
\]
\[
Q_{\lambda n_1} = \bigcap\{Q_{\lambda n_i} : i \in N\}.
\]

For each $\lambda \in \Lambda$, $m \in N$, let
\[
H(\lambda, m)_1 = P_{\lambda 1} \times g_{\lambda}^{-1}(Q_{\lambda m_1}), \quad H(\lambda, m)_2 = P_{\lambda 2} \times g_{\lambda}^{-1}(Q_{\lambda m_2}).
\]

Set
\[
\mathcal{H}_1 = \{H(\lambda, m)_1 : \lambda \in \Lambda \text{ and } m \in N\}
\]
and
\[
\mathcal{H}_2 = \{H(\lambda, m)_2 : \lambda \in \Lambda \text{ and } m \in N\}.
\]

We can easily introduce an equivalence relation $\sim$ on $X \times X(A)$ by the following: For each pair of points $p, q$ of $X \times X(A)$, $p \sim q$ if and only if $p \in C(q, \mathcal{H}_2)$. Let $Z[X]$ be the quotient space obtained from $X \times X(A)$ by $\sim$ with the quotient mapping $t : X \times X(A) \to Z[X]$. It is easy to check by (3) that
\[
C(p, \mathcal{H}_1) = C(p, \mathcal{H}_2) \quad \text{for each point } p \in X \times X(A).
\]

We construct a pair family $\mathcal{R}$ of $Z[X]$ as follows:
\[
\mathcal{R} = \bigcup\{\mathcal{R}(n_1, \ldots, n_k) : n_1, \ldots, n_k \in N, \ k \in N\},
\]
\[
\mathcal{R}(n_1, \ldots, n_k) = \{R(\lambda_1, \ldots, \lambda_k ; n_1, \ldots, n_k) : \lambda_1, \ldots, \lambda_k \in \Lambda, \ n_1, \ldots, n_k \in N, \ k \in N\},
\]
where
\[
R(\lambda_1, \ldots, \lambda_k ; n_1, \ldots, n_k) = (R_1(\lambda_1, \ldots, \lambda_k ; n_1, \ldots, n_k), R_2(\lambda_1, \ldots, \lambda_k ; n_1, \ldots, n_k))
\]
and for each $s = 1, 2$
\[
R_s(\lambda_1, \ldots, \lambda_k ; n_1, \ldots, n_k) = t\left(\bigcap\{P_{\lambda_i s} \times g_{\lambda_i}^{-1}(Q_{\lambda_i n_i}) : i \leq k\}\right), \quad \lambda_1, \ldots, \lambda_k \in \Lambda, \ n_1, \ldots, n_k \in N, \ k \in N.
\]

By (4), each $R_2(\lambda_1, \ldots, \lambda_k ; n_1, \ldots, n_k)$ is open in $Z[X]$, i.e., $\mathcal{R}$ is a pair family of $Z[X]$. It is easily seen that, for each $n_1, \ldots, n_k \in N, \ k \in N$, $\mathcal{R}(n_1, \ldots, n_k)$ is a $\sigma$-locally finite pair family in $Z[X]$. Thus $\mathcal{R}$ is so in $Z[X]$. To see that $\mathcal{R}$ is a pair network for $Z[X]$, let $[p] \in O$, where $p = (x, y) \in X \times X(A)$ and $O$ is open in $Z[X]$. Then there exists an open rectangle $U \times V$ of $X \times X(A)$ such that
\[
\{x\} \times K_p \subset U \times V \subset t^{-1}(O).
\]

Since $\bigcup\{\mathcal{R}_n : n \in N\}$ is a pair network for $X$, there exists $\lambda_1 \in \Lambda$ such that
\[
x \in P_{\lambda_1 1} \subset P_{\lambda_1 2} \subset U.
\]

Therefore, for some $m_1 \in N$
\[
C(p, \mathcal{H}_2) \subset H(\lambda_1, m_1)_1 \subset H(\lambda_1, m_1)_2 \subset U \times X(A).
\]
By virtue of (3) and (4), there exist $m_2, \ldots, m_k \in N$, $\lambda_2, \ldots, \lambda_k \in \Lambda$ such that

$$
C(p, \mathcal{R}) \subset \bigcap \{H(\lambda_i, m_i)_1 : i = 1, 2, \ldots, k\} \\
\subset \bigcap \{H(\lambda_i, m_i)_2 : i = 2, \ldots, k\} \subset X \times V.
$$

From both (4) and (6), we have

$$[p] \in R_1(\lambda_1, \ldots, \lambda_k : m_1, \ldots, m_k) \\
\subset R_2(\lambda_1, \ldots, \lambda_k : m_1, \ldots, m_k) \subset O.
$$

Hence $\mathcal{R}$ is a pair network for $Z[X]$.

We recall Worrell's method in [W] proving that if $Y$ is the image of a developable space under a closed mapping and if each point inverse of $Y$ has a meta-Lindelöf boundary in the domain, then $Y$ is a developable space. Applying essentially this method, we can easily prove that if $Y$ is a first countable image of a developable space under a closed mapping and if each point of $Y$ except a $\sigma$-discrete closed subset $Y_1$ has a compact point inverse, then $Y$ is a developable space. Let $g : Z[X] \to Y$ be a mapping such that $f = g \cdot t$. Then obviously $g$ is a closed mapping. By Lemma 3 $Y$ is decomposed as $Y = Y_1 \cup Y_0$, where $Y_1$ is a $\sigma$-discrete closed subset of $Y$ and, for each $y \in Y_0$, $f^{-1}(y)$ is compact in $X(A)$. This implies that $g^{-1}(y)$ is compact in $Z[X]$. Thus, by the above we can conclude that $Y$ is a developable space. This completes the proof.

**Remark.** By the same argument, we can prove that a first countable $\varphi$-extension of $\sigma$-spaces, that is, regular spaces having a $\sigma$-locally finite closed network in the sense of Okuyama [O], has a $\sigma$-locally finite closed network.

**References**


Joetsu University of Education, Joetsu, Niigata 943, Japan