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ON SUBADDITIVE FUNCTIONS

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ABSTRACT. The main result says that every one-to-one subadditive function $f\colon (0,\infty) \to (0,\infty)$ such that $\lim_{t\to 0} f(t) = 0$ must be continuous everywhere. A construction of a broad class of discontinuous subadditive bijections of $(0,\infty)$ which are bounded in every vicinity of 0 is given. Moreover, a problem of extension of a subadditive function defined in $(0,\infty)$ to a subadditive even function in $\mathbb{R}$ is considered

INTRODUCTION

It is well known that many local properties of a subadditive function depend on its behaviour in a neighbourhood of the origin (cf., e.g., Hille and Phillips [2], Rosenbaum [6], and Kuczma [3]). One of the most important results says that *if* $f\colon (0,\infty) \to \mathbb{R}$ *is subadditive, continuous at the origin, and* $f(0) = 0$ *then for every* $t > 0$ *there exist both one-sided limits* $f(t-)$, $f(t+)$ *and*

$$f(t+) \le f(t) \le f(t-).$$

In the first section of the present paper we prove that *every one-to-one subadditive function* $f\colon (0,\infty) \to (0,\infty)$ *such that* $\lim_{t\to 0} f(t) = 0$ *is continuous in* $(0,\infty)$. This improves the main result from our recent paper [4] where, using some properties of quasi-monotonic functions, we have proved that *every subadditive and right-continuous at* 0 *bijection of* $[0,\infty)$ *is a homeomorphism.* This result turns out to be very useful in applications. In particular, it allows us to give a characterization of the L^p-norms and to prove the converse of Minkowski's inequality under weak regularity conditions (cf. [4, 5]). Therefore, a detailed discussion of the assumptions of this theorem might be of some interest.

It is a natural question to ask whether the right continuity at 0 can be replaced by a weaker assumption of the boundedness in a neighbourhood of 0. In §2, using some new geometric criteria of subadditivity, we settle this problem in the negative. Actually we construct a whole class of subadditive bijections of $\mathbb{R}_+$ which are bounded in a neighbourhood of 0 and discontinuous at 0. Let us mention that earlier Ważka [7], trying to decide this problem, proved that the

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boundedness in a neighbourhood of 0 of the inverse f^{-1} of any subadditive bijection f of $\mathbb{R}_+$ implies the continuity of f^{-1} at 0. We present a short argument for this quite interesting fact.

In §3 we deal with the problem of the extension of a subadditive function in $(0, \infty)$ to an even subadditive function in $\mathbb{R}$. In particular, we prove that *if a subadditive function* $f: (0, \infty) \to \mathbb{R}$ *such that* $\lim_{t\to 0} f(t) = 0$ *admits an even subadditive extension* $F: \mathbb{R} \to \mathbb{R}$ *then it is continuous everywhere, nonnegative, and* $F(0) = 0$.

1. One-to-one subadditive functions in $(0, \infty)$

A function $f: (0, \infty) \to \mathbb{R}$ is said to be *subadditive* if

$$f(s+t) \le f(s) + f(t) \qquad (s, t > 0). \tag{1}$$

In the same way we define a subadditive function in $\mathbb{R}_+ := [0, \infty)$ or in $\mathbb{R}$. If the inequality is reversed, the function is termed superadditive.

Theorem 1. *If* $f: (0, \infty) \to \mathbb{R}_+$ *is one-to-one, subadditive, and*

$$\lim_{t\to 0} f(t) = 0 \tag{2}$$

then f *is continuous in* $(0, \infty)$. (*In particular,* f *is an increasing homeomorphism of* $(0, \infty)$ *onto the range of* f.)

Proof. From the restriction on the range of f we have $f \ge 0$. By (1) and (2) the function f is locally bounded above in $(0, \infty)$. Therefore, for every $t > 0$ the numbers

$$g := \liminf_{s\to t+} f(s), \qquad G = \limsup_{r\to t+} f(r)$$

exist and are finite. Hence there are the sequences r_n, s_n $(n \in \mathbb{N})$, such that

$$t < s_n < r_n \quad (n \in \mathbb{N}); \qquad \lim_{n\to\infty} r_n = t, \quad \lim_{n\to\infty} f(s_n) = g, \quad \lim_{n\to\infty} f(r_n) = G.$$

From (1) we have

$$f(r_n) \le f(s_n) + f(r_n - s_n) \qquad (n \in \mathbb{N}).$$

Letting n tend to infinity, by (2) we get $G \le g$. It means that the right limit $f(t+)$ exists. From (1), $f(r_n) \le f(t) + f(r_n - t)$ $(n \in \mathbb{N})$. Letting here $n \to \infty$, we obtain $f(t+) \le f(t)$. In a similar way we can prove that the left limit $f(t-)$ exists and $f(t) \le f(t-)$ (cf. [1, p. 248, Theorem 7.8.3] where measurability of f is assumed). Thus we have proved that for every $t > 0$

$$f(t+) \le f(t) \le f(t-). \tag{3}$$

Suppose for an indirect proof that f is discontinuous at a point $t > 0$, i.e., that $f(t+) < f(t-)$. It follows that there exist t_1, t_2 such that

$$0 < t_1 < t < t_2 \quad \text{and} \quad f(t_1) > f(t_2) > 0.$$

Put

$$A := \{t > 0 | s \in [t, t_1] \Rightarrow f(s) > f(t_2)\}.$$

We have $A \ne \varnothing$ because $t_1 \in A$. Let $t_0 := \inf A$. Suppose that $t_0 > 0$. Since f is one-to-one, only one of the following two cases could happen:

I. $f(t_0) > f(t_2)$ or
II. $f(t_0) < f(t_2)$.

Case I. Since $f(t_0-) \geq f(t_0) > f(t_2)$, there exists a $\delta > 0$ such that for all $s \in [t_0 - \delta, t_0]$ we have $f(s) > f(t_0)$. This contradicts the definition of t_0.

Case II. We have $f(t_0) < f(t_2)$. From (3) we get $f(t_0+) \leq f(t_0) < f(t_2)$. Hence there is a $\delta > 0$ such that for every $s \in [t_0, t_0+\delta]$ we have $f(s) < f(t_2)$. This also contradicts the definition of t_0.

Thus we have proved that $t_0 = 0$.

Now, from the definitions of the set A and t_0 we obtain

$$\lim_{s \to 0} f(s) \geq f(t_2) > 0,$$

which is the desired contradiction.

Remark 1. It follows from (3) that *every subadditive increasing function* $f\colon (0, \infty) \to \mathbb{R}_+$ *satisfying condition* (2) *must be continuous.*

Remark 2. For an arbitrary decreasing function $\phi\colon (0, \infty) \to \mathbb{R}$, the function $f\colon (0, \infty) \to \mathbb{R}$ given by the formula $f(t) = t\phi(t)$ $(t > 0)$ is subadditive. If, moreover, $\phi(t) < 0$ for all $t > 0$ then f is strictly decreasing and, consequently, one-to-one. It shows that, in Theorem 1, the assumption of f to be nonnegative is essential. Taking positive ϕ we obtain the following two equivalent statements.

Corollary 1. *Let* $\phi\colon (0, \infty) \to (0, \infty)$ *be decreasing.*

(a) *If the function* $(0, \infty) \ni t \to t\phi(t)$ *is one-to-one and*

$$\lim_{t \to 0} t\phi(t) = 0 \tag{4}$$

then ϕ *is continuous in* $(0, \infty)$.

(b) *If* ϕ *is discontinuous and* (4) *holds then the function* $(0, \infty) \ni t \to t\phi(t)$ *is not one-to-one.*

Remark 3. It is easily seen that every one-to-one subadditive function $f\colon (0, \infty) \to \mathbb{R}_+$ must be positive. Consequently the function f in Theorem 1 is actually of the type $f\colon (0, \infty) \to (0, \infty)$.

Remark 4. Since f is superadditive if and only if the function $(-f)$ is subadditive, one can reformulate all the above results for superadditive functions.

Remark 5. It is easy to find a lot of one-to-one (but not onto) discontinuous subadditive functions $f\colon (0, \infty) \to (0, \infty)$ which fail to satisfy condition (2): $\lim_{t \to 0} f(t) = 0$.

2. Subadditive bijections of $(0, \infty)$ and $\mathbb{R}_+$

Let us note that a function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a subadditive bijection if and only if $f(0) = 0$ and $f|_{(0, \infty)}$, the restriction of f to $(0, \infty)$, is a subadditive bijection of $(0, \infty)$.

Hence, as an obvious consequence of Theorem 1 we obtain

Corollary 2 (cf. [4]). *If* $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ *is one-to-one, onto, right continuous at* 0, *and subadditive then it is a homeomorphism.*

Thus every subadditive function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ being bijective and continuous at 0 must be continuous everywhere. On the other hand, all the hitherto known examples of the discontinuous subadditive bijections of $\mathbb{R}_+$ are *unbounded in every neighbourhood of* 0 (cf. [4, Examples 1 and 2]). In this section we prove

Theorem 2. *There exists a subadditive bijection of* $\mathbb{R}_+$ *which is bounded in a neighbourhood of* 0 *and discontinuous at* 0.

The proof of this theorem, in which we construct a broad class of such functions, consists of some auxiliary results.

Let $f\colon (0, \infty) \to (0, \infty)$ be a function and $\mathbf{P} \subset (0, \infty)\times(0, \infty)$ an arbitrary set. Lemma 1 and Lemma 2 give some simple conditions under which the implication

$$\mathrm{Graph}(f) \subset \mathbf{P} \Rightarrow f \text{ is subadditive}$$

holds true. Thus they may be interpreted as geometric criteria of subadditivity.

Lemma 1. *Let* a *and* b *be positive real. If the graph of a function* $f\colon (0, \infty) \to (0, \infty)$ *is contained in the set*

$$\mathbf{P} := \bigcup_{k=0}^{\infty} (ka, (k+1)a] \times ((k+2)b, (k+3)b]$$

then f *is subadditive and locally bounded.*

Proof. The graph of a function f is contained in $\mathbf{P}$ if and only if for every nonnegative integer k we have

$$ka < t \leq (k+1)a \Rightarrow (k+2)b < f(t) \leq (k+3)b. \tag{5}$$

(A geometric interpretation of this implication is visualized by a picture of the set $\mathbf{P}$ shown in Figure 1.) Take arbitrary positive s and t. There exist nonnegative integers j and k such that

$$ja < s \leq (j+1)a, \qquad ka < t \leq (k+1)a,$$

and, consequently,

$$(j+k)a < s+t \leq (j+k+2)a.$$

Now from (5) we get

$$f(s) > (j+2)b; \qquad f(t) > (k+2)b; \qquad f(s+t) \leq (j+k+4)b,$$

which shows that $f(s+t) \leq f(s) + f(t)$. The last statement is obvious.

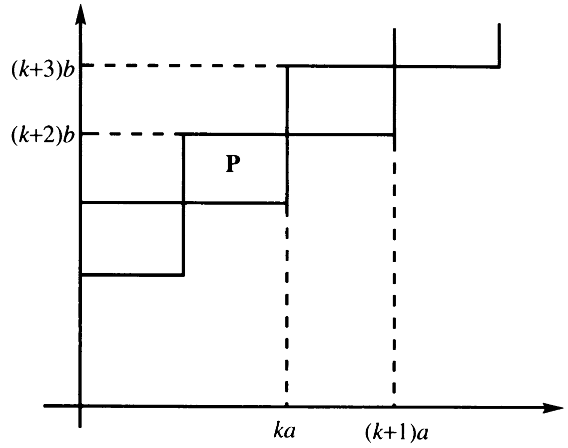

FIGURE 1

Lemma 2. *Let* $f_n\colon (0, \infty) \to (0, \infty)$, $n \in \mathbb{N}$, *be a nonincreasing sequence of subadditive functions, and let* $(a_n)_{n=0}^{\infty}$ *be an increasing sequence of real numbers such that* $a_0 = 0$ *and* $\lim_{n\to\infty} a = \infty$. *Then the function* $f\colon (0, \infty) \to (0, \infty)$ *given by the formula*

$$f(t) := f_n(t), \qquad a_{n-1} < t \le a_n \ (n \in \mathbb{N}), \tag{6}$$

is subadditive.

Proof. For arbitrary s, $t > 0$ there exist m, $n \in \mathbb{N}$ such that

$$a_{m-1} < s \le a_m, \qquad a_{n-1} < t \le a_n.$$

Putting $k := \max(m, n)$ we have $s + t > a_{k-1}$. Now (6) and the monotonicity of the sequence (f_n) imply

$$f(s+t) \le f_k(s+t) \le f_k(s) + f_k(t) \le f_m(s) + f_n(t) = f(s) + f(t),$$

which completes the proof.

Now using Lemmas 1 and 2 we can deduce the following crucial geometric test of subadditivity (compare Figure 2).

Lemma 3. *For fixed* $a > 0$, $b > 0$ *and a sequence of positive integers* $(m_n)_{n=1}^{\infty}$, *put*

$$a_1 := a, \quad a_{n+1} := (m_n + 1)a_n; \quad b_n := 2^{-n}b \qquad (n \in \mathbb{N})$$

and define the sets

$$\mathbf{S}_1 := \bigcup_{k=0}^{m_1 - 1} (ka_1, (k+1)a_1] \times ((k+2)b_1, (k+3)b_1],$$

$$\mathbf{S}_n := \bigcup_{k=1}^{m_n} (ka_n, (k+1)a_n] \times ((k+2)b_n, (k+3)b_n] \qquad (n \in \mathbb{N},\ n \ge 2).$$

If the graph of a function $f\colon (0, \infty) \to (0, \infty)$ *is contained in the set* $\mathbf{S} := \bigcup_{n=1}^{\infty} \mathbf{S}_n$ *then* f *is subadditive.*

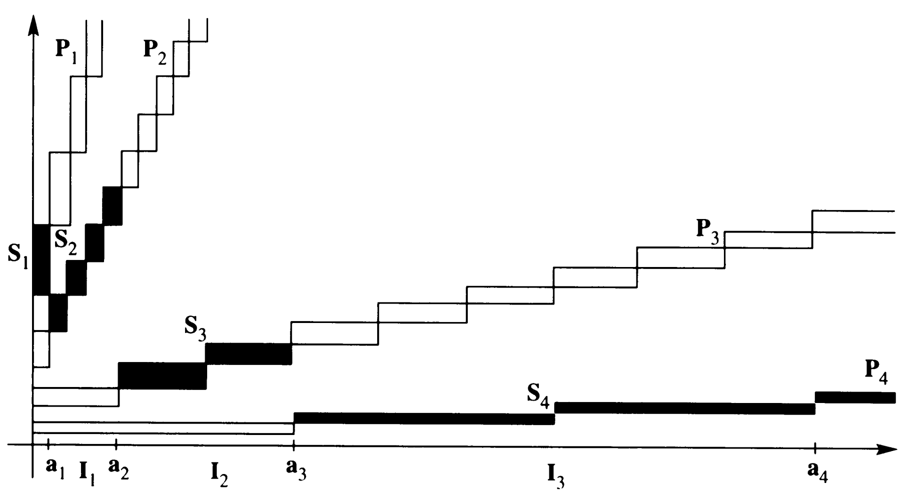

FIGURE 2

Proof. For every positive integer n the set $\mathbf{S}_n$ is a subset of

$$\mathbf{P}_n := \bigcup_{k=0}^{\infty} (ka_n, (k+1)a_n] \times ((k+2)b_n, (k+3)b_n]$$

(cf. Figure 2). Denote by I_n the projection of the set $\mathbf{S}_n$ into the first coordinate axis. Let $f_n\colon (0, \infty) \to (0, \infty)$ be an arbitrary function the graph of which is a subset of the set $\mathbf{P}_n$ and such that $f_n(t) = f(t)$ for all $t \in I_n$ $(n \in \mathbf{N})$. In view of Lemma 1 the functions f_n $(n \in \mathbf{N})$ are subadditive. Now it is easily seen that the sequence (f_n) and the function f satisfy all the assumptions of Lemma 2. This completes the proof.

Lemma 4. *For every interval* $(c, d]$ *there exist pairwise disjoint sets* $C_{i,j}$, $i \in \mathbb{N}$, $j = 1, 2, \ldots, 2^{i-1}$, *such that*

1°. *every* $C_{1,j}$ *is of the cardinality continuum*;

2°. $C_{i,j} \subset (c + (j-1)r_i, c + jr_1]$ *where* $r_i := 2^{1-i}(d-c)$; *and*

$$(c, d] = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{2^{i-1}} C_{i,j}.$$

Proof. We start the construction with $i = 2$. For $C_{2,1}$ and $C_{2,2}$ we take arbitrary sets of the cardinality continuum and of measure zero such that

$$C_{2,1} \subset (c, c + r_2] = \left(c, \frac{c+d}{2}\right], \qquad C_{2,2} \subset (c + r_2, c + 2r_2] = \left(\frac{c+d}{2}, d\right].$$

Now, for $i = 3$, there exist the sets $C_{3,j}$, $j = 1, 2, 3, 4$, of the cardinality continuum and of measure zero such that

$$C_{3,j} \subset (c + (j-1)r_3, c + jr_3], \qquad C_{3,j} \cap (C_{2,1} \cup C_{2,2}) = \varnothing.$$

Repeating this procedure we obtain the disjoint sets

$$C_{i,j} \subset [c + (j-1)r_i, c + jr_i), \qquad i \in \mathbb{N}, \ i \geq 2, \ j = 1, 2, \ldots, 2^{i-1},$$

each of measure zero and the cardinality continuum. To finish the proof it is enough to put

$$C_{1,1} := (c, d] \setminus \bigcup_{i=2}^{\infty} \bigcup_{j=1}^{2^{i-1}} C_{i,j}.$$

Corollary. *Let* $\mathbf{S}$ *be defined as in Lemma* 3. *If the sequence* $(2^{-n}m_n)_{n=1}^{\infty}$ *is increasing and unbounded then there exists a decomposition of the interval* $(0, \infty)$ *onto pairwise disjoint sets*

$$A_{1,k} \quad (k = 0, 1, \ldots, m_1 - 1); \qquad A_{n,k} \quad (n \in \mathbb{N}, \ n \geq 2, \ k = 1, \ldots, m_n)$$

of the cardinality continuum and such that $A_{n,k} \subset ((k+2)b_n, (k+3)b_n]$.

Proof. The set $\mathbf{S}$ is a union of the pairwise disjoint component rectangles of $\mathbf{S}$, i.e., maximal rectangles of the form $\mathbf{R} = (\alpha, \beta] \times (\gamma, \delta]$. For every such rectangle there is exactly one component rectangle $\mathbf{R}_0 = (\alpha_0, \beta_0] \times (c, d]$ such that

$$(c, d] \supseteq (\gamma, \delta], \qquad \mathbf{S} \cap (0, \alpha_0] \times (c, d] = \varnothing.$$

(If between a rectangle $\mathbf{R}$ and the second coordinate axis there are not any points of $\mathbf{S}$ then $\mathbf{R}_0$ coincides with $\mathbf{R}$; cf. Figure 3. The other case is shown

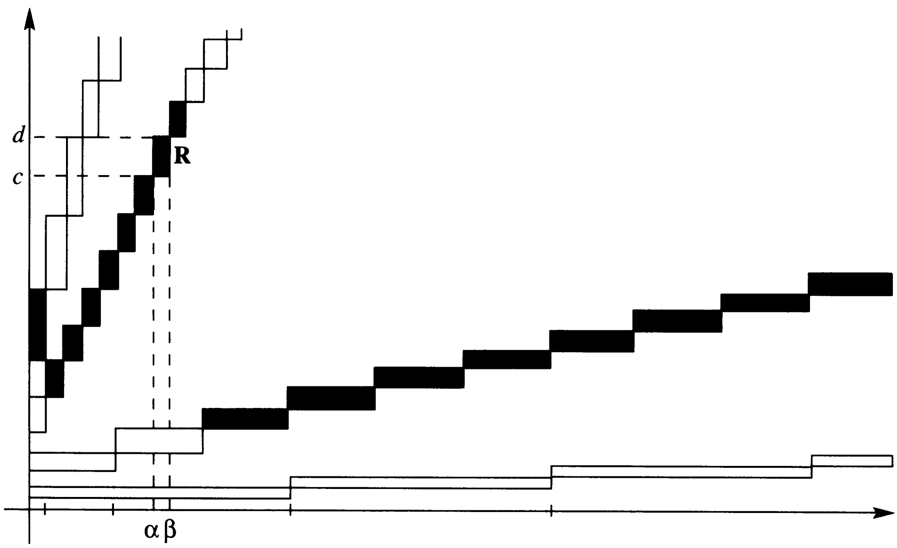

FIGURE 3

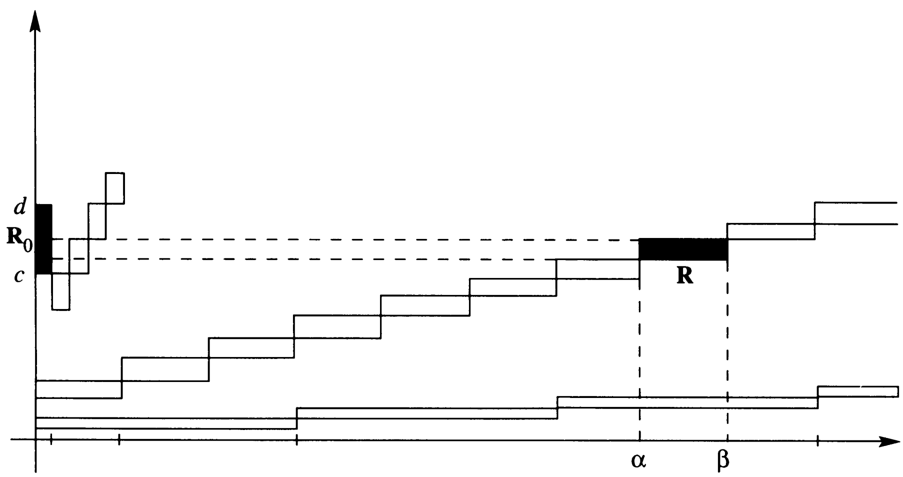

FIGURE 4

in Figure 4.) We say that $\mathbf{R}_0$ is the *left component rectangle* corresponding to a given component rectangle $\mathbf{R}$. Since the sequence $2^{-n}m_n$ $(n \in \mathbb{N})$ is increasing and unbounded, all the intervals $(c, d]$, the projections of the left rectangles on the second axis, form a disjoint covering of the interval $(0, \infty)$. Let $\mathbf{R} = (ka_n, (k+1)a_n] \times ((k+2)b_n, (k+3)b_n]$ be a fixed component rectangle of $\mathbf{S}$, and let $\mathbf{R}_0 = (\alpha_0, \beta_0] \times (c, d]$ be its left component rectangle. Then (by the definition of the numbers b_n in Lemma 3 and r_i in Lemma 4) there exist uniquely determined $i \in \mathbb{N}$ and $j \in \{1, \dots, 2^{i-1}\}$ such that

$$(\gamma, \delta] = (c + (j-1)r_i, c + jr_i], \qquad r_i := 2^{1-i}(d-c).$$

To finish the proof it is sufficient to put $A_{n,k} := C_{i,j}$ where $C_{i,j}$ is defined in Lemma 4.

Now Theorem 2 is an obvious consequence of the following proposition.

Proposition. *There exists a function* $f\colon (0, \infty) \to (0, \infty)$ *having the following properties*:

1°. f *is subadditive one-to-one and onto*;

2°. *for every* $c > 0$, f *is bounded in the interval* $(0, c)$ *and*

$$\inf\{f(t)\colon t \in (0, c)\} > 0.$$

Proof. In Lemma 3 take arbitrarily fixed $a > 0$, $b > 0$ and a sequence of positive integers $(m_n)_{n=1}^{\infty}$ such that the sequence $(2^{-n}m_n)_{n=1}^{\infty}$ is increasing and

$$\lim_{n\to\infty} 2^{-n}m_n = \infty.$$

(One could choose, for instance, $m_n := 3^n$.)

Note that the projections $(\alpha, \beta]$ on the first coordinate axis of the component rectangles $(\alpha, \beta] \times (\gamma, \delta]$ of the set $\mathbf{S}$ form a disjoint decomposition of the interval $(0, \infty)$. Therefore, to define the function f it is sufficient to define $f|_{(\alpha, \beta]}$, its restriction to the interval $(\alpha, \beta]$. Since

$$(\alpha, \beta] \times (\gamma, \delta] = (ka_n, (k+1)a_n] \times ((k+2)b_n, (k+3)b_n]$$

for some $n \in \mathbb{N}$ and $k \in \{1, \dots, 2^{n-1}\}$, we can define $f|_{(\alpha, \beta]}$ to be an arbitrary bijection of $(\alpha, \beta]$ onto the set $A_{n,k}$ defined in the corollary; this is possible because $A_{n,k}$ is of the cardinality continuum.

Since $A_{n,k} \subset (\gamma, \delta]$, the graph of $f|_{(\alpha, \beta]}$ is contained in $(\alpha, \beta] \times (\gamma, \delta] \subset \mathbf{S}$. It follows that the graph of f is contained in $\mathbf{S}$ and, by Lemma 3, f is subadditive. As the sets $A_{n,k}$ form a disjoint decomposition of $(0, \infty)$, the function f is a bijection of $(0, \infty)$ onto itself. This completes the proof.

Remark 6. If f is a function from the above proposition then $\liminf_{t\to 0} f(t) > 0$. On the other hand (cf. [4, Example 1]), there exist subadditive bijections f of $(0, \infty)$ such that

$$\liminf_{t\to 0} f(t) = 0 \quad \text{and} \quad \limsup_{t\to 0} f(t) = \infty.$$

In this connection a question arises of whether *every bounded in a vicinity of* 0 *subadditive bijection* $f\colon (0, \infty) \to (0, \infty)$ *such that*

$$\liminf_{t\to 0} f(t) = 0$$

is a homeomorphism of $(0, \infty)$. In the context of the next result this problem seems to be rather difficult to decide.

Theorem 3 (Ważka [7]). *If* $f\colon (0, \infty) \to (0, \infty)$ *is a subadditive bijection and* f^{-1} *is bounded in a neighbourhood of* 0 *then*

$$\lim_{t\to 0+} f^{-1}(t) = 0.$$

Proof. Put $g := f^{-1}$ and suppose that $g(t) < M$ for $0 < t < \delta$. We first show that

$$g(t) < M/2 \quad \text{for all } t < \delta/2.$$

In fact, if there were a $t_0 < \delta/2$ such that $s_0 := g(t_0) \geq M/2$, then, in view of the subadditivity of f, we would have

$$t_1 := f(2s_0) \leq 2f(s_0) = 2t_0 < \delta \quad \text{and} \quad g(t_1) = 2s_0 \geq M,$$

which is a contradiction. In the same way we show that

$$g(t) < M/2^n \quad \text{for all } t < \delta/2^n .$$

This completes the proof.

Remark 7. Note that the relevant theory for *superadditive* bijection of $(0, \infty)$ is very simple because every such function is strictly increasing and, consequently, homeomorphic.

3. On even extensions of subadditive functions defined in $(0, \infty)$

It is known that, in general, a subadditive function $f\colon (0, \infty) \to \mathbb{R}$ does not admit a finite subadditive extension in $\mathbb{R}$. It is the case for instance when f is positive, decreasing in $(0, \infty)$, and $f(0+) = \infty$ (cf. [2, p. 245]). It is also known that f has an odd subadditive extension if and only if f is additive (cf. [3, p. 402, Lemma 9]).

In this section we examine the functions $f\colon (0, \infty) \to \mathbb{R}$ which admit even subadditive extensions.

Theorem 4. *Suppose that* $f\colon (0, \infty) \to \mathbb{R}$ *is subadditive and*

$$\lim_{t \to 0} f(t) = 0 .$$

If f *admits an even subadditive extension* $F\colon \mathbb{R} \to \mathbb{R}$ *then* F *(as well as* f*) is continuous everywhere, nonnegative, and* $F(0) = 0$.

Proof. Since F is subadditive and even, we have, for all $t \in \mathbb{R}$, $t \neq 0$, that

$$0 \leq F(0) = F(t + (-t)) \leq F(t) + F(-t) = 2F(t) = 2f(|t|) .$$

Thus F is nonnegative and, letting $t \to 0$, we get $F(0) = 0$. Since F is also continuous at 0, for every $t \in \mathbb{R}$ there exist the one-sided limits $F(t-)$, $F(t+)$, and $F(t+) \leq F(t) \leq F(t-)$ (cf. [2, p. 248, Theorem 7.8.3]). Now the relation $F(t) = f(|t|)$, $t \neq 0$, implies that $F(t+) = F(t-)$. This completes the proof.

Theorem 5. *Every subadditive increasing function* $f\colon (0, \infty) \to \mathbb{R}$ *such that* $f(0+) = 0$ *is continuous and its even extension* F *with* $F(0) = 0$ *is subadditive in* $\mathbb{R}$.

Proof. From the monotonicity of f we have $f(t+) \leq f(t) \leq f(t-)$ for every $t > 0$. On the other hand, the subadditivity of f and $f(0+) = 0$ imply (3) (cf. Remark 1). Hence f is continuous. For convenience we may assume that $f(0) = 0$. For s, $t \in \mathbb{R}$ we have

$$F(s + t) = f(|s + t|) \leq f(|s| + |t|) \leq f(|s|) + f(|t|) = F(s) + F(t) ,$$

which was to be shown.

Example. It can easily be verified that the function $F\colon \mathbb{R} \to \mathbb{R}_+$ given by the formulas

$$F(t) = \begin{cases} |t| & \text{for } |t| \leq 2 , \\ 4 - |t| & \text{for } 2 < |t| \leq 3 , \\ 1 & \text{for } |t| > 3 \end{cases}$$

is subadditive, continuous, $F(0) = 0$ but not increasing in $(0, \infty)$. Thus the assumption of monotonicity of f in Theorem 5 is not necessary.

However, we can prove that this example is the best of its kind.

Theorem 6. *Suppose that* $F:\mathbb{R}\to\mathbb{R}$ *is subadditive, even, continuous at* 0, $F(0)=0$, *and* $F(\infty):=\lim_{t\to\infty}F(t)$ *exists. Then*

$$\sup F(\mathbb{R})\le 2F(\infty).$$

This inequality is the best of its kind.

Proof. By Theorem 4 the function F is nonnegative and continuous. Assume that $F(\infty)$ is finite (otherwise there is nothing to prove) and suppose, for an indirect proof, that $\sup F(\mathbb{R}) > 2F(\infty)$. It follows that $\sup F(\mathbb{R})$ is finite and there exists a point $t_0\ge 0$ such that $F(t_0)=\sup F(\mathbb{R})$. For an arbitrary sequence t_n $(n\in\mathbb{N})$ such that $t_n\to\infty$, put $s_n:=t_0-t_n$ $(n\in\mathbb{N})$. As F is subadditive and even we have

$$F(t_0)\le F(t_n)+F(t_0-t_n)=F(t_n)+F(t_n-t_0)\qquad (n\in\mathbb{N}).$$

Letting $n\to\infty$ we get $F(t_0)\le 2F(\infty)$, which is a contradiction. Since in the example we have $\sup F(\mathbb{R})=4$ and $F(\infty)=2$, the proof is complete.

Remark 8. Using a similar argument one can generalize Theorem 6 as follows. *If* $f:\mathbb{R}\to\mathbb{R}$ *is subadditive, even, continuous at* 0, $F(0)=0$ *then* $\sup F(\mathbb{R})\le \liminf_{t\to\infty}F(t)+\limsup_{t\to\infty}F(t)$ *and this inequality is the best of its kind.* To prove the last statement it is enough to consider the subadditive function $F:\mathbb{R}\to\mathbb{R}_+$, $F(t):=|\sin t|$.

Remark 9. Let I be an interval. A function $\phi: I\to\mathbb{R}$ is said to be subadditive if $\phi(s+t)\le\phi(s)+\phi(t)$ whenever s, t, and $s+t$ are in I.

If ϕ is subadditive in $I=[0,a]$ $(a>0)$ then it has a lot of subadditive extensions $f:\mathbb{R}_+\to\mathbb{R}$ and the individual properties of ϕ have no influence on those of f (cf., e.g., Lemma 1). Let us mention that Bruckner [1] proved that *any subadditive function* $\phi: I\to\mathbb{R}$ *has a unique maximal subadditive extension* $f:\mathbb{R}_+\to\mathbb{R}$ and showed that f inherits much of its behaviour from the behaviour of ϕ. Primarily Bruckner dealt with superadditive functions ϕ which are nonnegative showing, among other things, that the hereditary properties are continuity, Lipschitz continuity, and partially, but in a very interesting way, differentiability. Using the idea of the proof of Theorem 4 in paper [1] and applying our Theorem 5 it is not difficult to prove

Theorem 7. *Let* $\phi:[0,a]\to\mathbb{R}_+$ *be increasing, subadditive,* $\phi(0)=0$, *and suppose* $f:\mathbb{R}_+\to\mathbb{R}$ *is the maximal subadditive extension of* ϕ. *Then* $F:\mathbb{R}\to\mathbb{R}$ *given by* $F(t):=f(|t|)$ *is subadditive in* $\mathbb{R}$.

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