MINIMAL SURFACES AND $H$-SURFACES
IN NONPOSITIVELY CURVED SPACE FORMS

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Abstract. We show that if the Gauss curvature of a surface of constant mean curvature in a nonpositively curved space form is sufficiently pinched, the surface is stable. In this case, we also give an upper bound for the inradius. We then show that the inradius of a stable minimal surface in Euclidean space, which is contained in a solid cylinder, is bounded above by a constant depending only on the radius of the cylinder.

Let $M^3(c)$ denote a 3-dimensional oriented space form of constant sectional curvature $c \leq 0$. Let $X: M \to M^3(c)$ be a smooth immersion of a smoothly bounded surface $M$ with curvature $K$ and mean curvature $h$. Set

$$K = \max_M K, \quad \bar{K} = \min_M K.$$

We show

Theorem I. For $h, c \in \mathbb{R}$ with $-A^2 := h^2 + c \leq 0$, there exist universal constants $\omega(c, h) \geq e^2$ with the following property:

If $M \subset M^3(c)$ is a smooth orientable surface with constant mean curvature $h$ and

$$(-A^2 - K)/(-A^2 - \bar{K}) \leq \omega(c, h)$$

then $M$ is stable. In addition, $7.4\ldots = e^2 \leq \omega(0, 0) \leq 10.75\ldots$ holds.

Theorem II. If $-\infty < K, \bar{K} < -A^2$, and $M$ contains a geodesic ball $B_r(x_0)$ of radius $r$ then

$$r^2 \leq \frac{\pi^2}{4(-A^2 - \bar{K})} \log \left[(-A^2 - K)/(-A^2 - \bar{K})\right].$$

In the second part of this paper we consider surfaces in $\mathbb{E}^N$ which are extrinsically bounded in some way. Let $C_R = \{X = (X_1, X_2, X_3) \in \mathbb{E}^3 \mid x_1^2 + x_2^2 < R^2\}$. We show

Theorem III. There exists a constant $c_1 > 0$ with the following property: If $M \subset \mathbb{E}^3$ is an orientable stable minimal surface with $B_r(p) \subset M$ and $M \subset C_{R_1} \setminus C_{R_2}$ for some $R_1 > R_2 \geq 0$ then $r^2 \leq c_1(R_1^2 - R_2^2)$.

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Before beginning the proofs, we review some basic facts about surfaces in $M^3(c)$.

Let $M \rightarrow M^3(c)$ be an orientable surface. Let $ds^2$ denote the induced metric. $(M, ds^2)$ may be considered a Riemann surface in a natural way by introducing isothermal coordinates $(x, y)$ and using $z = x + iy$ as a complex coordinate. Doing so, $ds^2$ may be expressed as $ds^2 = e^{\rho}|dz|^2$ and the curvature is

$$K = -2e^{-\rho}\rho_{zz}.$$  

The second fundamental form of $M$ has an expression

$$\Pi = \text{Re}\{\phi dz^2 + he^{\rho}dz\,dz\}$$

where $\phi dz^2$ is an invariant quadratic differential and $h$ is the mean curvature. The fundamental equations of the immersion are those of Gauss

$$|\phi|^2 = e^{2\rho}(h^2 + c - K)$$

and Codazzi

$$\phi_z = e^{\rho}h_z.$$

When $h = \text{const}$, (3) implies that $\phi$ is holomorphic in $z$. It then follows that either $\phi \equiv 0$ and $M$ is totally umbilic or the zeros of $\phi$ are isolated.

**Lemma 1.** Let $M \subset M^3(c), c \leq 0$, have constant mean curvature $h$ such that $h^2 + c < 0$. Assume $M$ has no umbilics. Then the conformal metric

$$ds^2 = (h^2 + c-K)ds^2$$

has curvature $\tilde{K}$ satisfying

$$\tilde{K} \geq 1.$$  

**Proof.** Using (1)–(3) one has

$$0 = \Delta \log |\phi| = \Delta \log (-A^2 - K)^{1/2} + \Delta \rho$$

$$= \Delta \log (-A^2 - K)^{1/2} - 2K$$

where $\Delta = 4e^{-\rho}\partial z\partial \bar{z}$. Therefore, using (1) to compute $\tilde{K}$ one has

$$\tilde{K} = -(-A^2 - K)^{-1}\{\Delta \log (-A^2 - K)^{1/2} - K\} = (-A^2 - K)^{-1}(-K).$$

Since $K < 0$ on $M$, (5) follows.

**Proposition 1.** Again assume $h = \text{const}$, $h^2 + c \equiv -A^2 \leq 0$, and

$$0 \leq -A^2 - \tilde{K} \leq -A^2 - K \leq -A^2 - K < \infty.$$  

Then the first (nontrivial) eigenvalue $\lambda_1$ of the problem

$$\begin{cases}
\Delta \psi + 2\lambda(-A^2 - K)\psi = 0 & \text{on } M, \\
\psi = 0 & \text{on } \partial M
\end{cases}$$

satisfies

$$\left[\frac{1}{2}\log((-A^2 - K)/(-A^2 - \tilde{K}))\right]^{-1} \leq \lambda_1.$$
Proof. Let $\psi \geq 0$ be the eigenfunction corresponding to $\lambda_1$. Let $g(x, y)$ denote the positive Green’s function of $M$. Then
\[
\psi(x) = \lambda_1 \int_{\Omega} (-2A^2 - 2K(y))\psi(y)g(x, y) \ast 1(y)
\]
\[
\leq \lambda_1 \int_{\Omega} -2K(y)\psi(y)g(x, y) \ast 1(y).
\]
Therefore,
\[
|\psi(y)| \leq \lambda_1 \|\psi\|_\infty \int -2K(y)g(x, y) \ast 1(y) = \lambda_1 \|\psi\|_\infty \nu(x),
\]
where $\nu$ solves
\[
\Delta \nu = 2K \quad \text{in } M,
\]
\[
\nu \equiv 0 \quad \text{on } \partial M.
\]
Choosing $x$ where $\psi$ achieves its maximum we arrive at
\[
1 \leq \lambda_1 \|\nu\|_\infty.
\]
By (6) and (9)
\[
\Delta(\nu - \log(-A^2 - K)^{1/2}) = 0 \quad \text{in } M
\]
and on $\partial M$
\[
\nu - \log(-A^2 - K)^{1/2} \leq -\log(-A^2 - K)^{1/2}.
\]
It follows from the maximum principle and (8) that on $M$
\[
\nu \leq \log(-A^2 - K)^{1/2} - \log(-A^2 - K)^{1/2}
\]
\[
\leq \log(-A^2 - K)^{1/2} - \log(-A^2 - K)^{1/2}.
\]
Using this and (10), (8) follows.

Proof of Theorem I. The surface $M$ is stationary for the functional
\[
J = \text{area} + 2H(\text{enclosed 3-volume}).
\]
The second variation of $J$ for variations of the form $\psi \cdot N$ where $N$ is the unit normal to $M$ and $\psi \in C_0^\infty(M)$ is given by $\delta^2 J = \int -\psi L\psi$. Here $L$ is the selfadjoint elliptic operator $L\psi = \Delta\psi + 2(-2A^2 - K)\psi$. Assuming the hypothesis of the theorem, we have by Proposition 1, $\lambda_1 \geq 1$. Using integration by parts we obtain
\[
\delta^2 J = \int -\psi L\psi = \int (|\nabla\psi|^2 - 2(-2A^2 - K)\psi^2)
\]
\[
\geq \int (|\nabla\psi|^2 - 2(-A^2 - K)\psi^2) \geq 0
\]
and $M$ is stable.

The upper bound for $\omega(0, 0)$ follows from Example I.

Remark. We have shown that under the hypothesis of Theorem I, the second variation of $J$ is nonnegative for all compactly supported variations. When $h \neq 0$ this is stronger than the condition that $\delta^2 J$ be nonnegative for all volume-preserving variations (cf. [B-DC]).
Example I. Let $C \subset \mathbb{R}^3 \approx \mathbb{C} \times \mathbb{R}$ be the catenoid parameterized by $X(u, v) = (e^u \cosh(u), u), (u, v) \in \mathbb{R} \times [0, 2\pi)$. One easily computes that the curvature is given by $K = -(\cosh u)^{-4}$ and that the support function is given by $s = -1 + u \tanh(u)$.

Denote by $\Omega_t$ the symmetric “waist” domain of the catenoid given by $|u| < t$. Then $\Omega_t$ will be stable as long as $s$ is negative, that is, for $t < t_1 \approx 1.2 \ldots$. For $\Omega_t$,

$$e^2 \geq \frac{K/K}{(\cosh t)^4}$$

holds for $t < t_1 \approx 1.0850 \ldots$. It follows that Theorem I has correctly predicted stability in this case. Furthermore, for $\Omega_{t_1}$, $K/K \approx 10.75 \ldots$ furnishes the upper bound in the corollary. Finally the total curvature of $\Omega_t$ is

$$2\pi \int_{-t_1}^{t_1} \cosh^{-2} u \, du \approx 2\pi(1.590 \ldots) > 2\pi.$$

Consequently the criteria of Theorem I is independent of the Barbosa-DoCarmo result [B-DC1].

To prove Theorem II we state without proof a special case of an eigenvalue estimate due to Gage [G].

Theorem (Gage). Let $\tilde{B}_r$ be a geodesic ball of radius $r$ contained in a surface of curvature $K \geq -\beta^2 = \text{const}$. Then the first Dirichlet eigenvalue of the Laplacian $\tilde{\Delta}$ on $\tilde{B}_r$ satisfies

$$\tilde{\lambda}_1 \leq \frac{\pi^2}{r^2} + \frac{\beta^2}{4}.$$  

Proof of Theorem II. For a region $\Omega \subset M$ let $\tilde{\lambda}_1(\Omega)$ denote the first Dirichlet eigenvalue of $\tilde{\Delta}$ for $\Omega$. Here $\tilde{\Delta}$ is the Laplacian for the metric $d\tilde{s}$ of Lemma 1. Let $\lambda_1(\Omega)$ be the first eigenvalue of the problem

$$\Delta \psi + 2\lambda(-A^2 - K) \psi = 0 \quad \text{in } \Omega,$$

$$\psi = 0 \quad \text{on } \partial \Omega.$$  

Since $\tilde{\Delta} = (-A^2 - K)^{-1}\Delta$, it is clear that $\tilde{\lambda}_1(\Omega) = 2\lambda_1(\Omega)$. Let $\gamma$ be a minimizing geodesic of length $\tilde{r}$ for the metric $d\tilde{s}$. Then

$$\tilde{r} = \int_{\gamma} (-A^2 - K)^{1/2} \, ds \geq (-A^2 - K)^{1/2} \int_{\gamma} ds.$$

It follows that $\tilde{B}_{\tilde{r}(-A^2 - K)^{1/2}} \subset B_r$. By a well-known monotonicity property of eigenvalues

$$\tilde{\lambda}_1(\tilde{B}_{\tilde{r}(-A^2 - K)^{1/2}}) \geq \tilde{\lambda}_1(B_r) = 2\lambda_1(B_r).$$

So by Lemma 1 and Gage's Theorem with $\beta = 0$, we have

$$\frac{\pi^2}{r^2(-A^2 - K)} \geq 2\lambda_1(B_r).$$

Combining this with the lower bound of Proposition 1 yields the result.

We now consider surfaces in $\mathbb{E}^N$ which are extrinsically bounded.
Lemma 2. Let $M \subset \mathbb{R}^3$ be a minimal surface. Let $\Omega \subset M$ be a smoothly bounded subdomain, and let $\mu_1$ be the first Dirichlet eigenvalue of the Laplacian in $\Omega$. Assume

\begin{equation}
M \subset C_{R_1} \setminus C_{R_2}, \quad R_1 > R_2 > 0.
\end{equation}

Then

\begin{equation}
\mu_1 \geq \frac{2}{R_1^2 - R_2^2}.
\end{equation}

Proof. Define $\tau = \frac{1}{2}(x_1^2 + x_2^2)$. Let $N = (N_1, N_2, N_3)$ be a unit normal defined on a neighborhood in $M$. Then

\begin{align*}
\Delta \tau &= \frac{1}{2} \sum_{i=1,2} (2x_i \Delta x_i + 2\|\nabla x_i\|^2) = \|\nabla x_1\|^2 + \|\nabla x_2\|^2 \\
&= 1 - N_1^2 + 1 - N_2^2 = 1 + N_3^2 \geq 1.
\end{align*}

By (14) we have

\begin{equation}
\frac{R_2^3}{2} < \tau < \frac{R_1^3}{2}.
\end{equation}

Let $\psi \geq 0$ be a solution of $\Delta \psi + \mu \psi = 0$ with $\psi \equiv 0$ on $\partial \Omega$. Then

\begin{equation}
\psi(x) = \mu_1 \int_{\Omega} \psi(y) g(x, y) * 1(y)
\end{equation}

and consequently

\begin{equation}
|\psi(x)| \leq \mu_1 \|\psi\|_\infty \int_{\Omega} g(x, y) * 1(y).
\end{equation}

Taking $x$ values where $\psi$ achieves its maximum yields

\begin{equation}
1 \leq \mu_1 \max_{x \in \Omega} \int_{\Omega} g(x, y) * 1(y) = \mu_1 \max_{x \in \Omega} S(x),
\end{equation}

where $S$ solves

\begin{align*}
\Delta S &= -1 \quad \text{in } \Omega, \\
S &= 0 \quad \text{on } \partial \Omega.
\end{align*}

Therefore,

\begin{equation}
\Delta (S + \tau) \geq 0 \quad \text{in } \Omega \quad \text{and} \quad S + \tau = \tau \leq R_1^2/2 \quad \text{on } \partial \Omega.
\end{equation}

By the maximum principle

\begin{equation}
S \leq \frac{R_1^2}{2} - \tau \leq \frac{R_1^2}{2} - \frac{R_2^2}{2} \quad \text{in } \Omega.
\end{equation}

Combining this with (17) proves (15).

Proof of Theorem III. Since $M$ is stable, it follows by a result of Schoen [Sc, Corollary 4] that there is an estimate $K(x) \geq -2\alpha/r^2$, for all $x \in B_{r/2}$, where $\alpha$ is a universal constant. Using this lower bound for $K$ in Gage’s upper bound for $\mu_1$ gives

\begin{equation}
\mu_1 \leq \frac{\alpha}{4r^2} + \frac{\pi^2}{r^2} =: \frac{4C_1}{r^2}.
\end{equation}

Combining this with the lower bound for $\mu_1$ in Lemma 2 gives the result.
Corollary. Let $M \subset \mathbb{E}^3$ be a complete minimal surface. If there exists $p \in M$ such that

\begin{equation}
\limsup_{r \to \infty} \frac{1}{r} \int_{B_r(p)} (-K) = 0
\end{equation}

then $M$ is not contained in a cylinder.

Proof. Assume to the contrary that $M \subset C_R$ for some $R$, $0 < R < \infty$. Let $r_0$ be a constant with $r_0^2 > c_1 R^2$ with $c_1$ as in Theorem III. Then any disc $B_{r_0} \subset M$ is unstable. By a result of Barbosa and DoCarmo [B-DC1] $\int_{B_{r_0}} (-K) > 2\pi$. Taking the sequence $r_n = nr_0$, one finds that since $B_{r_n}(p)$ contains at least $n$ disjoint geodesic balls of radius $r_0$,

\begin{equation}
\frac{1}{r_n} \int_{B_{r_n}(p)} (-K) > \frac{2\pi n}{nr_0} = \frac{2\pi}{r_0} \gg 0,
\end{equation}

giving a contradiction.

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References


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