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FINITE MOMENTS PERTURBATIONS OF $y'' = 0$ IN BANACH ALGEBRAS

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ABSTRACT. Rigorous asymptotics for a basis of $y''+g(x)y=0$, $x\in[1,+\infty)$, is derived in the framework of Banach algebras. The key assumption is $\int_1^{+\infty} x^k\|g(x)\|\,dx<\infty$ for $k=1$ *or* $k=2$. Such results improve and generalize previous work on linear second-order matrix differential equations.

1. INTRODUCTION

In this paper we are concerned with linear second-order differential equations like

$$y''+g(x)y=0, \qquad x\in[1,+\infty), \tag{1}$$

where the functions g and y take values in a given real or complex *Banach algebra* $\mathscr{B}$, with a unit element e, $g\in C^0([1,+\infty);\mathscr{B})$, and g is "asymptotically small" in the sense that the first or the second moment of $\|g\|$ is finite.

An important special class is encountered when g and y are $n\times n$ matrices. This case is relevant to the asymptotic theory of linear second-order *systems* of differential equations. Previous work on this subject, mainly motivated by the investigation on nonoscillation properties of solutions, only concerned the case of a *symmetric* or *Hermitian* matrix coefficient g (cf. [1, 2, 7]). Under these hypotheses and the finiteness of the first moment of $\|g\|$, it was shown that (1) has a *recessive* solution like $u(x)=I+E(x)$, where I denotes the $n\times n$ identity matrix, and the "error term" $E(x)$ can be estimated explicitly [1, 2]. For the second (dominant) solution v, however, only the *qualitative* behavior $v(x)\sim xI$, $x\to+\infty$, could be established.

The goal of the present paper is to obtain asymptotic approximations with *precise error bounds* for a basis of the (right) $\mathscr{B}$-module of solutions to (1). In fact, it is easily seen that such a module is free and has rank 2, by using Hille's theory for first-order equations (cf., e.g., [4, Chapter 6]). Besides the generalization of classical results to the framework of Banach algebras, that can be of *finite* as well as of *infinite* dimension, either commutative or not, in this

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paper we shall remove the restriction of symmetry for g (indeed, the algebra is not required to be involutory). Moreover, when the second moment of $\|g\|$ is finite, we are able to obtain an asymptotic representation with an *error bound also* for a dominant solution, thus improving, in particular, the available results for the matrix case.

In [9] we studied recently similar problems for real scalar differential equations like (1), along with their discrete analogue (i.e., second-order linear difference equations). The spirit was that of complementing Olver's rigorous asymptotic results of the Liouville-Green (or WKBJ) type, valid in the case of g *not* "asymptotically small" [5]. So far the only available contribution to the Liouville-Green theory for the *matrix* case seems to be that of [8].

The key technique we employ below consists of obtaining differential equations satisfied by the error terms in the representations for u, v. Such equations are then converted into Volterra integral equations whose solutions are estimated asymptotically by successive approximations. The integrals involved are interpreted, in general, in the sense of Bochner [3, Chapter 3].

2. The asymptotic theorem

In this section we shall prove the main theorem, for which it is useful to introduce the functions

$$m_k(x) := \int_x^{+\infty} t^k \|g(t)\|\, dt, \qquad k = 0, 1, 2. \tag{2}$$

These functions are well defined (and are actually infinitesimal as $x \to +\infty$) whenever the corresponding moments, $m_k(1)$, exist.

Theorem 2.1. *Consider the linear second-order differential equation* (1), *with* g *and* y *taking values in a given Banach algebra* $\mathscr{B}$, *with unit element* e, *and* $g \in C^0([1, +\infty); \mathscr{B})$. *Suppose that*

$$\int_1^{+\infty} t\|g(t)\|\, dt < \infty. \tag{3}$$

Then the right $\mathscr{B}$*-module of solutions to* (1) *is generated by the pair* $(u(x), v(x))$, *with*

$$u(x) = e + \varepsilon(x), \qquad v(x) = x(e + \eta(x)), \tag{4}$$

where

$$\|\varepsilon(x)\| \le \exp\{m_1(x)\} - 1, \qquad \|\varepsilon'(x)\| \le m_0(x)\exp\{m_1(x)\}, \tag{5}$$

and $\eta(x) = o(1)$ *as* $x \to +\infty$.

When the stronger condition

$$\int_1^{+\infty} t^2\|g(t)\|\, dt < \infty \tag{6}$$

replaces (3), *there exists a second solution of* (1) *of the form*

$$w(x) = xe + \omega(x), \tag{7}$$

where

$$\begin{aligned} &\|\omega(x)\| \le m_2(x)\exp\{m_1(x)\}, \\ &\|\omega'(x)\| \le m_1(x) + m_0(x)m_2(x)\exp\{m_1(x)\}, \end{aligned} \tag{8}$$

replacing $v(x)$ *in the pair* $(u(x), v(x))$.

Proof. Assuming that (3) holds, and looking for a solution of the form $u(x) = e + \varepsilon(x)$, equation (1) yields the "error equation"

$$\varepsilon'' + g(x)[e + \varepsilon] = 0, \qquad x \in [1, +\infty). \tag{9}$$

Now, it is easily verified that every C^2-solution to the integral equation

$$\varepsilon(x) = \int_x^{+\infty} (x - t)g(t)[e + \varepsilon(t)]\,dt \tag{10}$$

satisfies (9). The integral in (10) is intended in the sense of *Bochner* [3, Chapter 3], and all limits and differentiations can be interchanged with integration according to the generalized version of the dominated convergence theorem (cf. [3, Theorem 3.7.9, p. 83]). Introduce recursively the sequence

$$\begin{aligned} &h_0(x) := 0, \\ &h_{s+1}(x) := \int_x^{+\infty} (x - t)g(t)[e + h_s(t)]\,dt, \qquad s = 0, 1, 2, \ldots. \end{aligned} \tag{11}$$

It is immediate to show by induction on s that such a sequence is well defined and $h_s \in C^2([1, +\infty); \mathscr{B})$, in view of (3). Next we shall prove that the series

$$\varepsilon(x) := \sum_{s=0}^{\infty} [h_{s+1}(x) - h_s(x)] \tag{12}$$

converges *uniformly* in $[1, +\infty)$. In fact, the estimate

$$\begin{aligned} \|h_{s+1}(x) - h_s(x)\| &\le \frac{[m_1(x)]^{s+1}}{(s+1)!} \\ &\le \frac{[m_1(1)]^{s+1}}{(s+1)!}, \qquad s = 0, 1, 2, \ldots, \quad x \in [1, +\infty), \end{aligned} \tag{13}$$

can be proved again by induction on s. Details are standard (cf. [1], e.g., for the matrix case). From (12) and (13) then the first estimate in (5) follows. As

$$h'_{s+1}(x) = \int_x^{+\infty} g(t)[e + h_s(t)]\,dt,$$

we get

$$\|h'_{s+1}(x) - h'_s(x)\| \le m_0(x)\frac{[m_1(x)]^s}{s!},$$

and hence the second estimate in (5).

A similar procedure for the second derivatives leads to the estimate

$$\|h''_{s+1}(x) - h''_s(x)\| \le \|g(x)\|\frac{[m_1(x)]^s}{s!},$$

which shows that $\varepsilon \in C^2([1, +\infty); \mathscr{B})$, and $\|\varepsilon''(x)\| \le \|g(x)\| \exp\{m_1(x)\}$. Finally, writing by (11), (12)

$$\begin{aligned} \varepsilon(x) &= h_1(x) + \sum_{s=1}^{\infty} \int_x^{+\infty} (x - t)g(t)[h_s(t) - h_{s-1}(t)]\,dt \\ &= h_1(x) + \int_x^{+\infty} (x - t)g(t)\varepsilon(t)\,dt, \end{aligned}$$

we see that the function $\varepsilon(x)$ in (12) indeed solves (10).

Now we look for a second solution to (1) in the form

$$v(x) = u(x) \int_{x_0}^{x} f(t)\, dt, \tag{14}$$

where $x_0 \in [1, +\infty)$ and the $\mathscr{B}$-valued function f have to be determined. Differentiating twice, we obtain

$$v'' = -gv + 2u'f + uf'.$$

Now in view of the first inequality of (5), $u(x)$ is invertible in $\mathscr{B}$ for all

$$x > x_1 := \inf\{x : x \geq 1,\ \|\varepsilon(x)\| < 1\}$$

(cf. [6, Theorem 10.7, p. 231]). Then, v satisfies (1) *iff* f solves the *first-order* differential equation

$$f' = -2u^{-1}u'f, \qquad x > x_1. \tag{15}$$

By a theorem of Hille [4, Theorem 6.4.4, p. 227], there exists in $[x_0, +\infty)$, $x_0 > x_1$, a solution to (15) such that $f \overset{\mathscr{B}}{\to} e$ as $x \to +\infty$, provided that $\|(u(x))^{-1}u'(x)\| \in L^1([x_0, +\infty))$. This condition is fulfilled if

$$\int_{x_0}^{+\infty} \|u'(x)\|\, dx = \int_{x_0}^{+\infty} \|\varepsilon'(x)\|\, dx < \infty,$$

i.e.,

$$\int_{x_0}^{+\infty} m_0(x)\, dx = \int_{x_0}^{+\infty} \left(\int_{x}^{+\infty} \|g(t)\|\, dt \right) dx < \infty,$$

which is true by Fubini's theorem in view of (3). Finally, we get from (14) that

$$\mathscr{B}\text{-}\lim_{x\to+\infty} \frac{v(x)}{x} = \mathscr{B}\text{-}\lim_{x\to+\infty} \frac{1}{x} \int_{x_0}^{x} f(t)\, dt = e, \tag{16}$$

as $f \overset{\mathscr{B}}{\to} e$, in view of the generalized L'Hôpital's rule as proved in [10] (cf. also a well-known Abelian theorem in [3, Theorem 18.2.1, p. 505]). Therefore (4) has been proved.

Suppose now that (6) holds. In this case a second solution, say $w(x)$, can be constructed of the form (7), (8). Again, an equation for the error term $\omega(x)$ is obtained,

$$\omega'' + g(x)[xe + \omega] = 0,$$

which is satisfied by every C^2-solution of

$$\omega(x) = \int_{x}^{+\infty} (x - t) g(t)[te + \omega(t)]\, dt.$$

Proceeding similarly to the case of the first solution by successive approximations, we obtain $w(x)$ as in (7), (8) and the additional estimate $\|\omega''(x)\| \leq \|g(x)\| m_2(x) \exp\{m_1(x)\}$. Notice that, in (8), $\|\omega'(x)\| = O(m_1(x))$ as $x \to +\infty$.

The last thing to be proved is that the pair $(u(x), v(x))$ in (4) [or $(u(x), w(x))$] is *a basis* for the right $\mathscr{B}$-module of solutions to (1). Such a module is free and has rank 2. Introducing the *Wronskian matrix*

$$\mathbf{W}(x) := \begin{pmatrix} u(x) & v(x) \\ u'(x) & v'(x) \end{pmatrix}, \qquad \mathbf{W}(x) \in M_2(\mathscr{B}), \tag{17}$$

we have that $(u(x), v(x))$ is a basis for (1) *iff* $\mathbf{W}(x)$ is *invertible* for every $x \in [1, +\infty)$.

Now it is easily proved, from their asymptotic behavior, that $u(x)$ and $v(x)$ are linearly independent solutions to (1) and therefore that the linear operator $\mathbf{W}(x)$ is *injective* for each fixed x. In the special case $\mathscr{B} = M_n(\mathbf{R})$ or $\mathscr{B} = M_n(\mathbf{C})$, it follows immediately that $\mathbf{W}(x)$ is invertible. This is *not* true, however, in the general case.

Here is a direct proof of the fact that indeed $\mathbf{W}(x)$ is invertible in a neighborhood of $+\infty$ (and hence everywhere). Notice first that from (14), (16), and (5) follows $x\varepsilon'(x) = o(1)$, and thus

$$v'(x) = e + o(1), \qquad x \to +\infty.$$

Splitting $\mathbf{W}(x)$ as

$$\mathbf{W}(x) = \begin{pmatrix} e + \varepsilon(x) & x[e + \eta(x)] \\ 0 & e + o(1) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \varepsilon'(x) & 0 \end{pmatrix} =: \mathbf{W}_0(x) + \mathbf{W}_1(x),$$

it is clear that the first summand, $\mathbf{W}_0(x)$, is invertible in $M_2(\mathscr{B})$ for x sufficiently large, as such are its diagonal elements in view of [6, Theorem 10.7, p. 231]. Then, denoting by $\mathbf{I}$ the unit element of the Banach algebra $M_2(\mathscr{B})$,

$$\mathbf{W}(x) = \mathbf{W}_0(x)[\mathbf{I} + (\mathbf{W}_0(x))^{-1}\mathbf{W}_1(x)].$$

Therefore, it suffices to show that

$$|||(\mathbf{W}_0(x))^{-1}\mathbf{W}_1(x)||| < 1$$

for x sufficiently large; the (multiplicative) norm $|||\cdot|||$ in $M_2(\mathscr{B})$ is that canonically induced by $\|\cdot\|$. This is true since we get by easy calculations

$$(\mathbf{W}_0(x))^{-1}\mathbf{W}_1(x) = \begin{pmatrix} -x[e + o(1)]\varepsilon'(x) & 0 \\ [e + o(1)]^{-1}\varepsilon'(x) & 0 \end{pmatrix},$$

in a neighborhood of $+\infty$, where it is clear that all entries are $o(1)$ as $x \to +\infty$. The proof that $(u(x), w(x))$ is also a basis (when (6) holds) is completely analogous.

3. Remarks and examples

Some remarks are now in order.

Remark 3.1. It is obvious, first of all, that equations like $y'' + yg(x) = 0$, under the hypotheses (3) or (6), can be treated similarly. *Left* $\mathscr{B}$-modules are involved in this case.

Remark 3.2. We stress that the Wronskian matrix introduced in §2, while it seems to be the natural choice in view of discussing that a given pair of independent solutions generates the whole module, differs from that adopted in [1, 2, 7] for the matrix case. The latter definition preserves *certain* properties of the scalar case but involves the *adjoint* matrix. Moreover, in [1, 2, 7] it is *required* that the matrix coefficient $g(x)$ be *Hermitian*.

Remark 3.3. Whenever the Banach algebra is a C^*-algebra and $g(x)$ is Hermitian, the second solution to (1) as given by (14) can be *explicitly* represented in terms of the first solution as

$$v(x) = u(x) \int_{x_0}^{x} (u(t))^{-1}[u^*(t)]^{-1}\, dt. \tag{18}$$

In fact, one can easily check that $f(x) = (u(t))^{-1}[u^*(t)]^{-1}$ solves (15) *and* $f \overset{\mathscr{B}}{\to} e$. If, in a general Banach algebra, the property $g(t)g(s) = g(s)g(t)$ holds for all t, s sufficiently large, the second solution in (14) has the form

$$v(x) = u(x) \int_{x_0}^{x} (u(t))^{-2}\, dt. \tag{19}$$

Finally, we give some simple examples, for the purpose of illustration.

Example 3.4. Assuming that $g(x) = O(x^{-p})$ with $p > 3$, that is, $\|g(x)\| \le Kx^{-p}$ for some $K > 0$, it is easy to show by Theorem 2.1 that

$$\begin{aligned} u(x) &= e + O\left(\frac{x^{2-p}}{p-2}\right), \qquad w(x) = xe + O\left(\frac{x^{3-p}}{p-3}\right), \\ u'(x) &= O\left(\frac{x^{1-p}}{p-1}\right), \qquad w'(x) = e + O\left(\frac{x^{2-p}}{p-2}\right) + O\left(\frac{x^{4-2p}}{(p-1)(p-3)}\right) \end{aligned} \tag{20}$$

(cf. [9] for the scalar case). Note, as in [9], the *double asymptotic* nature with respect to both the independent variable x and the parameter p. When $2 < p \le 3$, the representations for $u(x)$, $u'(x)$ in (20) still hold true.

Example 3.5. Suppose that $g(x) = ax^{-p}$, where a is a constant element in $\mathscr{B}$ and $p > 3$. Then the series in (12) leads to the representation

$$u(x) = e + h_s(x) + R_s(x), \qquad s = 0, 1, 2, \ldots, \tag{21}$$

where

$$\begin{aligned} h_s(x) &= \sum_{r=1}^{s} (-1)^r a^r c_r(p) x^{(2-p)r}, \\ c_r(p) &= \{(p-1)(p-2)(2p-3)(2p-4) \\ &\quad \cdots [rp - (2r-1)](rp - 2r)\}^{-1}, \qquad r = 1, 2, 3, \ldots, \end{aligned} \tag{22}$$

and the remainder can be estimated, via (12), (13), by

$$\|R_s(x)\| \le \frac{\exp\{m_1(1)\}}{(s+1)!} [m_1(x)]^{s+1} \tag{23}$$

(cf. [9]). Similar expansions (with bounds) can be derived for $v(x)$, as well as for $u'(x)$, $v'(x)$. If $2 < p \le 3$, all considerations concerning $u(x)$, $u'(x)$ are still valid. Observe that all these series (and, in general, (12)) represent the so-called *Liouville-Neumann expansions* for solutions to second-order linear differential equations [5] *in* the context of *Banach algebras.*

This method could be applied to other much more involved instances, e.g., $g(x) = \sum_{j=1}^{m} a_j x^{-p_j}$, with $a_j \in \mathscr{B}$ and $p_j > 3$ for all j. In such cases *symbolic manipulations* (computer algebra techniques) could be useful. The latter example includes in a natural way the matrix case of $\mathbf{g}(x) = \{a_{ij} x^{-p_{ij}}\}_{i,j=1}^{n}$ where the a_{ij} are real or complex constants and $p_{ij} > 3$ for $i, j = 1, 2, \ldots, n$. In fact, it suffices to split $\mathbf{g}(x) = \sum_{i,j} a_{ij} \mathbf{I}_{ij} x^{-p_{ij}}$, where $\mathbf{I}_{ij}$ is the $n \times n$ matrix whose only nonzero entry is in the (i, j) place.

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