COMPLETE MINIMAL SURFACES
AND THE PUNCTURE NUMBER PROBLEM

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Abstract. Given a nonnegative integer \( g \), let \( \mathcal{P}(g) \) denote the set of integers \( N \) such that an arbitrary compact Riemann surface with genus \( g \) can be completely conformally and minimally immersed in \( \mathbb{R}^3 \) (with finite total curvature) with exactly \( N \) punctures. We prove that the infimum of \( \mathcal{P}(g) \) is at most \( 4g \) and that the set \( \mathcal{P}(g) \) may not miss any \( 3g \) consecutive integers larger than the infimum of \( \mathcal{P}(g) \).

1. Introduction

A conformal immersion from a Riemann surface \( M \) to \( \mathbb{R}^3 \) is said to be minimal if its component functions are harmonic, and the maximum principle for harmonic functions prohibits any compact minimal surface. The tangential Gauss map of a conformal minimal immersion \( \varphi: M \to \mathbb{R}^3 \) is the map \( \Phi: M \to G(3, 2) \) taking \( p \in M \) to the (negatively) oriented tangent plane \( T_pM \subset 

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Our main result is the following

**Theorem.** (a) $4 \in \mathcal{P}(1)$,
(b) $\inf \mathcal{P}(g) \leq 4g$,
(c) $\mathcal{P}(g)$ may not miss any $3g$ consecutive integers larger than $\inf \mathcal{P}(g)$.

Note that (c) implies that $|\mathcal{P}(g)| = \infty$. That is to say, there are infinitely many admissible puncture numbers for any genus.

It should be remarked that the puncture number problem is closely related to the moduli problem for complete minimal surfaces. In fact, Xiaokang Mo recently constructed a complex space parametrizing the set of complete minimal surfaces with a fixed number of punctures using an earlier result of the author [Y1] on the puncture number problem. For details of Mo’s construction see [M].

2. An estimate lemma and the proof of the Theorem

To prove the Theorem we will need the following

**Lemma.** Let $M_g$ denote a compact Riemann surface of genus $g > 0$. Also let $F \in H^0(M_g, \mathcal{M}^*)$ be a not identically zero meromorphic function on $M_g$, and put

\[-d = d_F = \text{the degree of } F = \text{the total number of poles of } F,\]
\[-m = m_F = \text{the number of distinct zeros of } dF.\]

(i) There exists a conformal minimal immersion, complete with respect to the induced metric, $M_g \setminus \Sigma_F \to \mathbb{R}^3$, where the puncture set $\Sigma_F$ satisfies
\[-(*) \quad n + m \leq |\Sigma_F| \leq 2n + d + 3g - 2.\]

(ii) Suppose $D$ is any divisor on $M_g$ satisfying the following conditions:
\[-\deg D = -g, \quad D^+ = (dF)_0, \quad D^- \geq (dF)_\infty,\]
\[-\text{support } D^- = \text{support}(dF)_\infty,\]
where $D = D^+ - D^-$ with $D^+ > 0$. Also let $G \in L(-D)$, and put
\[-m' = \text{the number of distinct zeros of } G.\]

Then there exists a complete conformal minimal immersion $M_g \setminus \Sigma_F, G \to \mathbb{R}^3$
with
\[-(**) \quad |\Sigma_{F, G}| = n + m'.\]

**Proof.** For some $b_i \in \mathbb{Z}^+$ and distinct points $p_i \in M_g$ we have
\[-(F)_\infty = \sum_{i=1}^n b_ip_i.\]

Then $d = \sum b_i = \deg(F)_\infty$. Consider the meromorphic 1-form $dF$. We have
\[-(dF)_\infty = \sum (b_i + 1)p_i.\]
For some \( a_j \in \mathbb{Z}^+ \) and distinct points \( q_j \in M_g \) we have

\[
(dF)_0 = \sum_{j=1}^{m} a_j q_j.
\]

Since \((dF) = (dF)_0 - (dF)_{\infty}\) is a canonical divisor, its degree is \(2g-2\) so that

\[
\sum a_j = (2g-2) + n + d.
\]

Let \( D = D^+ - D^- \in \text{Div}(M_g) \) be as above. We then have

\[
D^+ = \sum_{j=1}^{m} a_j q_j, \quad D^- = \sum_{i=1}^{n} c_i p_i,
\]

where the \( c_i \)'s are some positive integers satisfying the conditions

\[
c_i \geq b_i + 1; \quad \sum c_i = 3g - 2 + n + d.
\]

Consider the complex vector space

\[
L(-D) = \{ \phi \in H^0(M_g, \mathcal{M}^*) : (\phi) \geq D \} \cup \{0\}.
\]

Since the support of \( D^+ = (dF)_0 \) is nonempty, we note that nonzero constant functions may not lie in \( L(-D) \). By the Riemann-Roch theorem

\[
\dim L(-D) = \deg(-D) - g + 1 + \dim L((dF) + D)
\]

\[
= 1 + \dim L((dF) + D) \geq 1.
\]

Thus there are nonconstant meromorphic functions belonging to \( L(-D) \). Let \( G \) be such a function, and also let \( m' \) denote the number of distinct zeros of \( G \). Using an argument totally similar to the one given in [Y1, pp. 708–710] we can find a complete conformal minimal immersion \( M_g \setminus \Sigma \to \mathbb{R}^3 \), where

\[
\Sigma = \text{support}(F)_{\infty} \cup \text{support}(G)_0.
\]

But \( F \) has \( n \) distinct poles and \( G \) has \( m' \) distinct zeros. This establishes the equality in \((**\)). Since \((G)_0 \geq D^+ = \sum_{j=1}^{m} a_j q_j \), we must have \( m' \geq m \), proving the first inequality in \((*) \). On the other hand, \( m' \) is at most equal to the degree of \( D^- \): the degree of \( D \) is negative and \( G \) can have at most \( \deg(D^-) \) many simple zeros. So

\[
m' \leq \sum c_i = 3g - 2 + n + d.
\]

The rest follows easily. \( \square \)

**Proof of Theorem.** We first consider the case \( g = 1 \). Let \( M_1 \) be any complex torus, and also let \( p(z) \) denote the Weierstrass function on it, where \( z \) is the (global) Euclidean coordinate on \( M_1 \). Let \( p \in M_1 \) be the lattice point so that

\[
(p)_{\infty} = 2p.
\]

Now \( dp = p' dz \), and it is well known that \( p' \) has three distinct simple zeros, say \( q_1, q_2, \) and \( q_3 \). Thus

\[
(dp) = (q_1 + q_2 + q_3) - 3p.
\]
Taking $F = p$ in the Lemma, we find that $d = 2, n = 1, m = 3$. Now take
\[ D = (q_1 + q_2 + q_3) - 4p. \]
Then $G = p' \in L(-D)$ and $m' = 3$. It follows that $|\Sigma_{F,G}| = 4$, proving (a) of the Theorem. We now assume that $g > 1$. We will show that we can pick $F \in H^0(M_g, \mathcal{M}^*)$ so that
\begin{equation}
2n_F + d_F + 3g - 2 \leq 4g. \tag{\dagger}
\end{equation}
Then the last inequality in (\ast) will give (b) of the Theorem. Let $p \in M_g$ be any Weierstrass point. This means that the gap sequence at $p$ is not $\{1, 2, \ldots, g\}$. In particular, there is a nongap $\tilde{d} \leq g$ at the point $p$, meaning that there is a meromorphic function $\tilde{F}$ on $M_g$ with $(\tilde{F})_{\infty} = \tilde{d}p$. Take $F = \tilde{F}$. Then $d = \tilde{d}, n = 1$, and (\dagger) follows. Before proving (c) in general we first look at the case $g = 1$. We have the following

**Observation.** Let $M_1$ be any complex torus, and also let $\hat{n} \geq 2$ be an integer. Then we can find a principal divisor on $M_1$ of the form
\[ \sum p_i - \sum q_i, \quad 1 \leq i \leq \hat{n}, \]
where the $p_i$'s and the $q_i$'s are distinct points of $M_1$.

The above observation is proved easily from Abel's theorem, which in this case states that a degree zero divisor on $M_1$ is principal if and only if the group sum (in the abelian group $M_1$) of the points of $D$ is zero.

Given $M_1$ and any $n = \hat{n} \geq 2$, we take $F$ such that
\[ (F) = \sum_{i=1}^{n} (p_i - q_i), \quad \text{the } p_i \text{'s and } q_i \text{'s are distinct.} \]

Then (\ast) gives
\begin{equation}
n + 1 \leq |\Sigma_F| \leq 3n + 1 \tag{1}
\end{equation}
since $d = n = \hat{n}$ and $m \geq 1$. On the other hand, we can take $F$ to be such that $(F)_{\infty} = dp$, where $p$ is any point of $M_1$ and $d$ is any integer $\geq 2$. Then $n = 1$, and the Lemma gives
\begin{equation}
2 \leq |\Sigma_F| \leq d + 3. \tag{2}
\end{equation}
The inequalities in (1), (2) prove (c) for the case $g = 1$. Now let $M_g$ denote a Riemann surface of genus $g > 1$. Then, given any $d > g$, there is a meromorphic function $F$ with $(F)_{\infty} = dp, \ p \in M_g$; just take $p$ to be any non-Weierstrass point. Then $n = 1$ and the Lemma gives
\begin{equation}
|\Sigma_F| \leq 3g + d. \tag{3}
\end{equation}
On the other hand, given any $n \in \mathbb{Z}^+$, there is a meromorphic function $F$ on $M_g$ with $n$ distinct poles (just add $n$ "gap functions" at $n$ distinct points). So given any $n \in \mathbb{Z}^+$ we can have
\begin{equation}
n + 1 \leq |\Sigma_F|. \tag{4}
\end{equation}
Combining (3), (4) we obtain the following
Proposition. Given any Riemann surface $M_g$ of genus $g$ and any integer $d > g$, there exists a complete conformal minimal immersion $M_g \setminus \Sigma \to \mathbb{R}^3$ with

$$d + 1 \leq |\Sigma| \leq 3g + d.$$ 

The above proposition easily implies (c) of the Theorem. □

3. Concluding remarks

One suspects that $\mathcal{P}(g) \supset \{N \in \mathbb{Z} : N \geq 4g\}$—a sort of deformation argument might work. On the other hand, it is unclear that $4g$ is actually the minimum of $\mathcal{P}(g)$. For example, one can reduce the degree of the meromorphic function used in producing the lower bound $4g$: on any $M_g$ it is well known that there is a meromorphic function of degree $d \leq (g + 3)/2$. However, it seems difficult to determine the number of distinct poles of such a meromorphic function.

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References


