GROWTH CONDITIONS FOR THIN SETS
IN VILENKIN GROUPS OF BOUNDED ORDER

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(Communicated by J. Marshall Ash)

Abstract. Let $G$ be a Vilenkin group of bounded order and $H_n$ a sequence of clopen subgroups of $G$ forming a base at the identity. If $E$ is a subset of $G$, let $N_n(E)$ denote the number of cosets of $H_n$ which intersect $E$. If

$$\lim_{n \to \infty} \frac{N_n(E)}{\log[G:H_n]} < \infty,$$

then $E$ is a U-set in the group $G$. It is also shown that for $G$ satisfying a growth condition and $\varphi(n) \to \infty$, there is an M-set, $E$, with

$$N_n(E) = O(\varphi(n) \log[G:H_n]).$$

Let $G$ be a Vilenkin group, i.e., a compact, totally disconnected, metric group, and let $H_n$ be a decreasing sequence of open subgroups of $G$ forming a neighborhood base at the identity of $G$. If $\Gamma$ is the Pontryagin dual of $G$, then for every complex valued function $\{c_\gamma\}$ on $\Gamma$, it is possible to form the "partial sums of the trigonometric series"

$$S_n(c, x) = \sum_{\gamma \in H_n^1} c_\gamma \gamma(x).$$

We say that a subset $E$ of $G$ is a set of uniqueness (U-set) if the only function $\{c_\gamma\}$ which vanishes at infinity on $\Gamma$ and whose partial sums converge to 0 off $E$ is the zero series. In [1] the author showed that for closed sets this is equivalent to the statement that $E$ supports no pseudofunctions. If a set is not a U-set, it is called an M-set.

In 1972, Kaufman [4] showed in the classical setting that if a subset of the circle intersects $O(\log(1/\varepsilon))$ intervals of length $\varepsilon$, then that subset is a U-set. The first goal of this paper is to show that an analogous result holds for Vilenkin groups of bounded order.

The second goal of this paper is to obtain a result similar to one of Kahane (see [3]). In Kahane's theorem, it is shown that there are M-sets on the circle intersecting $O(\varphi(\varepsilon) \log(1/\varepsilon))$ intervals of length $\varepsilon$ as long as $\varphi(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

For a subset $E$ of $G$, let $N_n(E)$ denote the number of cosets of $H_n$ that intersect $E$.

Received by the editors July 23, 1991 and, in revised form, January 13, 1992 and February 26, 1992.

1991 Mathematics Subject Classification. Primary 42C25; Secondary 43A46.

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0002-9939/93 $1.00 + .25 per page

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Theorem 1. Let $G$ be a group that is of bounded order. If $E$ is a subset of $G$ with

$$
\lim_{n \to \infty} \frac{N_n(E)}{\log[G : H_n]} < \infty,
$$

then $E$ is a set of uniqueness.

Proof. Choose $\varepsilon > 0$ so that $\gamma \in \Gamma$ and $|\gamma(x) - 1| < \varepsilon$ implies $\gamma(x) = 1$. This is possible since $G$ is of bounded order. Without loss of generality, we will assume that $N_n(E) < C \log[G : H_n]$ for all $n \geq 0$. Now choose $r$ to be an integer such that $r > 2C \log(\pi/\sin^{-1}(\varepsilon/2))$. For each $n$, let $L_n$ be a finite set with $L_nH_n = EH_n$ and with $\text{card}(L_n) < C \log[G : H_n]$. Break $L_n$ into $r$ disjoint subsets $L_{j,n}$ of approximately equal size. There are at most $Cr^{-1} \log[G : H_n]$ elements in each $L_{j,n}$. But

$$
\left(\frac{\pi}{\sin^{-1}(\varepsilon/2)}\right)^{-C} < [G : H_n]^{1/2} < [G : H_n].
$$

Since there are at most $\pi/\sin^{-1}(\varepsilon/2)$ elements of the unit circle such that all distances are more than $\varepsilon$, there are $\gamma_1, \gamma_2 \in H_n^\perp$ such that $|\gamma_1(x) - \gamma_2(x)| < \varepsilon$ for every $x \in L_{j,n}$.

In fact, letting $\gamma = \gamma_2 \gamma_1^{-1}$, we see that the cardinality of the set

$$
\Gamma_{j,n} = \{\gamma \in H_n^\perp : |\gamma(x) - 1| < \varepsilon \text{ for all } x \in L_{j,n}\}
$$

is at least $[G : H_n]^{1/2}$ and so is unbounded for each $j$ as $n \to \infty$. If $(\gamma_1, \ldots, \gamma_p) \in \Gamma_{1,n} \times \cdots \times \Gamma_{r,n}$, then since each $\gamma_j \in H_n^\perp$ and $EH_n = L_nH_n$,

$$
E \subseteq \bigcup_{j=1}^r \{x : |\gamma_j(x) - 1| < \varepsilon\}.
$$

By our choice of $\varepsilon$, for such $(\gamma_1, \ldots, \gamma_p)$, we have

$$
E \subseteq \bigcup_{j=1}^r \ker(\gamma_j).
$$

For each $p \geq 1$, we choose particular $(\gamma_1, \ldots, \gamma_r)$ as follows. Choose $n$ large enough so that $\text{card}\Gamma_{j,n} \geq (2^r - 1) \text{card} H_p^\perp$.

Pick any $\gamma_1 \in \Gamma_{1,n}$. If $\gamma_1, \ldots, \gamma_{j-1}$ have been chosen, pick

$$
\gamma_j \in \Gamma_{j,n} \setminus \bigcup_{i=1}^{j-1} \gamma_1^{e_1} \gamma_2^{e_2} \cdots \gamma_{j-1}^{e_{j-1}} H_p^\perp
$$

where the union is over all sequences $e_1, \ldots, e_{j-1}$ whose terms are all $-1$ or $0$. Because the cardinality of $\Gamma_{j,n}$ is large, such a choice is possible. The upshot is that no character of the form $\gamma_1^{e_1} \cdots \gamma_r^{e_r}$, where each $e_j$ is $0$ or $1$, can be in $H_p^\perp$ unless each $e_j = 0$. Now define

$$
f_p = \prod_{j=1}^r (1 - \gamma_j).$$
Then \( f_p = 0 \) on the neighborhood \( EH_n \) of \( E \) and \( (1-f_p)\hat{\chi}(x) = 0 \) if \( x \in H_p^+ \). If \( \mathcal{S} \) is any pseudofunction supported on \( E \), we have \( \mathcal{S}(f_p) = 0 \), so

\[
|\hat{\mathcal{S}}(1)| \leq (2^r - 1) \sup |\hat{\mathcal{S}}(\chi)|
\]

where the supremum is taken over those \( \chi \notin H_p^+ \). Since \( r \) is fixed and the supremum goes to 0 as \( p \to \infty \), \( \hat{\mathcal{S}}(1) = 0 \). By a standard argument, \( \mathcal{S} = 0 \). Thus \( E \) is a U-set. \( \square \)

It was previously known that if \( N_n(E) = o(\log[G : H_n]) \), then \( E \) is a Dirichlet set and hence a strong U-set \([2]\).

Now we turn to the question of the converse: how slow can \( N_n(E) \) grow for an M-set? For the classical case, \([3]\) shows that if \( \varphi(\epsilon) \to \infty \) as \( \epsilon \to 0 \) there are sets in the circle group such that \( N_n(E) = O(\varphi(\epsilon) \log(1/\epsilon)) \) and yet \( E \) fails to be a U-set. To get this result, Kahane used the properties of Brownian motion. For the corresponding result in the current setting, we have to produce an analog of Brownian motion on Vilenkin groups. I would like to thank Robert Kaufman for his suggestions at this point.

Let \( \xi_{xH_n} \) be a random point with values in \( H_n \) which is uniformly distributed with respect to normalized Haar measure on \( H_n \). We will assume that different \( \xi_{xH_k} \) are independent random points. Thus, if \( y \in \Gamma \), we have

\[
E[\gamma(\xi_{xH_n})] = \begin{cases} 1 & \text{if } y \in H_n^+, \\ 0 & \text{otherwise}. \end{cases}
\]

Notice that there is a separate random point for each coset of each \( H_n \). We now define a random function on \( G \) by

\[
\psi(x) = \sum_{n=0}^{\infty} \xi_{xH_n}.
\]

Notice that this sum converges for every point in \( G \) and that for two points \( x, y \in G \), if \( xH_n = yH_n \) then \( \psi(x)H_n = \psi(y)H_n \). Thus \( \psi \) enjoys a type of Lipschitz continuity. Thus we have for a subset \( E \) of \( G \),

\[
N_n(\psi(E)) \leq N_n(E).
\]

Now we present a converse to Theorem 1.

**Theorem 2.** Let \( G \) be a Vilenkin group. Assume that there is a constant \( C \) such that

\[
\sum_{n=0}^{\infty} \frac{[G : H_{n+1}]}{[G : H_n]^C} < \infty.
\]

Let \( \varphi(n) \uparrow \infty \) as \( n \to \infty \). Then there exists an M-set \( E \) in \( G \) such that

\[
N_n(E) = O(\varphi(n) \log[G : H_n]).
\]

**Proof.** Let \( \theta(n) = \varphi(n) \log[G : H_n] \). Let \( k_n \) be integers such that

\[
\theta(n) \leq k_1 k_2 \cdots k_n \leq 2\theta(n).
\]

Pick \( k_n \) points of \( H_n \) in distinct cosets of \( H_{n+1} \) and let \( \mu_n \) be a discrete measure on these points giving equal measure to each of them. Let \( \mu = \bigotimes_{n=1}^{\infty} \mu_n \). Then \( \mu \) is a measure supported on a set \( E \) with

\[
N_n(E) \leq 2\theta(n) \quad \text{and} \quad \mu(xH_n) \leq 1/\theta(n) \quad \text{for all } n \geq 0.
\]
Now consider the random measure $\nu$ where
\[ \int f d\nu = \int f \circ \psi d\mu . \]

For $\gamma \in H_{n+1}^\perp \setminus H_n^\perp$, we have
\[ \hat{\nu}(\gamma) = \int_G y \circ \psi d\mu = \sum_{xH_n} \int_{xH_n} y \circ \psi d\mu = \sum_{xH_n} \int_{xH_n} \prod_{k=0}^{\infty} \gamma(\xi_{xH_k}) d\mu(y) \]
\[ = \sum_{xH_n} \int_{xH_n} \prod_{k=0}^{n} \gamma(\xi_{xH_k}) d\mu(y) = \sum_{xH_n} \mu(xH_n) \prod_{k=0}^{n} \gamma(\xi_{xH_k}) . \]

Notice that for $y \in xH_n$, it follows that $\xi_{xH_k} = \xi_{yH_k}$ for all $k \leq n$. For each coset $xH_n$, we introduce the random variable
\[ Y_{xH_n} = \mu(xH_n) \prod_{k=0}^{n} \gamma(\xi_{xH_k}) . \]

Then $|Y_{xH_n}| = \mu(xH_n)$, and by taking conditional expectations with respect to all the $\xi_{xH_n}$ for this fixed $n$, we have that the $Y_{xH_n}$ are conditionally independent and
\[ E[Y_{xH_n} | \{\xi_{xH_n} : x \in G\}] = 0 . \]

Thus, using a sub-Gaussian inequality, we have for real $t$,
\[ E[\exp(t \Re \hat{\nu}(\gamma)) | \{\xi_{xH_n} : x \in G\}] \]
\[ = E \left[ \exp \left( \sum_{xH_n} t \Re Y_{xH_n} \right) | \{\xi_{xH_n} : x \in G\} \right] \]
\[ = \prod_{xH_n} E[\exp(t \Re Y_{xH_n}) | \{\xi_{xH_n} : x \in G\}] \leq \prod_{xH_n} \exp(t^2 \mu(xH_n)^2) \]
\[ = \exp \left( t^2 \sum_{xH_n} \mu(xH_n)^2 \right) \leq \exp \left( \frac{t^2}{\theta(n)} \right) . \]

Now we have
\[ E[\exp(t \Re \hat{\nu}(\gamma))] \leq \exp \left( \frac{t^2}{\theta(n)} \right) . \]

A similar expression holds for $\Im \hat{\nu}(\gamma)$. Recalling the definition of $\theta(n)$, we see
\[ P[\Re \hat{\nu}(\gamma) > \varphi(n)^{-1/3}] \leq \exp \left( -\frac{1}{4} \varphi(n)^{1/3} \log[G : H_n] \right) \]
\[ \leq \frac{1}{[G : H_n] \varphi(n)^{1/3} / 4} . \]

This holds for each $\gamma \in H_{n+1}^\perp \setminus H_n^\perp$, so
\[ P[\exists \gamma \in H_{n+1}^\perp \setminus H_n^\perp \text{ such that } \Re \hat{\nu}(\gamma) > \varphi(n)^{-1/3}] < \frac{[G : H_{n+1}]}{[G : H_n] \varphi(n)^{1/3} / 4} . \]
Since similar expressions hold for $-\text{Re} \tilde{\nu}(\gamma)$ and $\text{Im} \tilde{\nu}(\gamma)$ in place of $\text{Re} \tilde{\nu}(\gamma)$, and since $\varphi(n) \to \infty$, an application of the Borel-Cantelli Lemma along with (8) shows that $\lim \tilde{\nu}(\gamma) = 0$ almost surely. □

Notice that all Vilenkin groups of bounded order satisfy the condition in (8). In fact, any group with growth like $\exp(\exp(n))$ satisfies this condition.

Using the methods above, we may prove the following result.

**Theorem 3.** Let $G$ be a Vilenkin group satisfying the condition in (8). Given $\delta > 0$ there is a subset $E$ of $G$ such that $N_n(E) = O(\log[G : H_n])$ and a measure $\nu$ supported on $E$ with $\lim_{\gamma \to \infty} |\tilde{\nu}(\gamma)| \leq \delta$, where the limit is taken as $\gamma \to \infty$ in $\Gamma$. By Theorem 2 of [2], we see that $E$ is not a Dirichlet set.

**Proof.** In the previous proof, simply let $\varphi(n)$ be a constant where $\varphi(n)^{1/3} > 4C$ and $\varphi(n)^{-1/3} < \delta$. □

**Acknowledgment**

I would like to thank Robert Kaufman of the University of Illinois for suggestions leading to substantial improvements in Theorems 2 and 3.

**References**


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