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# GROWTH CONDITIONS FOR THIN SETS IN VILENKIN GROUPS OF BOUNDED ORDER

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ABSTRACT. Let $G$ be a Vilenkin group of bounded order and $H_n$ a sequence of clopen subgroups of $G$ forming a base at the identity. If $E$ is a subset of $G$, let $N_n(E)$ denote the number of cosets of $H_n$ which intersect $E$. If

$$\varliminf \frac{N_n(E)}{\log[G:H_n]} < \infty,$$

then $E$ is a U-set in the group $G$. It is also shown that for $G$ satisfying a growth condition and $\varphi(n) \to \infty$, there is an M-set, $E$, with

$$N_n(E) = O(\varphi(n)\log[G:H_n]).$$

Let $G$ be a Vilenkin group, i.e., a compact, totally disconnected, metric group, and let $H_n$ be a decreasing sequence of open subgroups of $G$ forming a neighborhood base at the identity of $G$. If $\Gamma$ is the Pontryagin dual of $G$, then for every complex valued function $\{c_\gamma\}$ on $\Gamma$, it is possible to form the "partial sums of the trigonometric series"

$$S_n(c, x) = \sum_{\gamma \in H_n^\perp} c_\gamma \gamma(x).$$

We say that a subset $E$ of $G$ is a set of uniqueness (U-set) if the only function $\{c_\gamma\}$ which vanishes at infinity on $\Gamma$ and whose partial sums converge to $0$ off $E$ is the zero series. In [1] the author showed that for closed sets this is equivalent to the statement that $E$ supports no pseudofunctions. If a set is not a U-set, it is called an M-set.

In 1972, Kaufman [4] showed in the classical setting that if a subset of the circle intersects $O(\log(1/\varepsilon))$ intervals of length $\varepsilon$, then that subset is a U-set. The first goal of this paper is to show that an analogous result holds for Vilenkin groups of bounded order.

The second goal of this paper is to obtain a result similar to one of Kahane (see [3]). In Kahane's theorem, it is shown that there are M-sets on the circle intersecting $O(\varphi(\varepsilon)\log(1/\varepsilon))$ intervals of length $\varepsilon$ as long as $\varphi(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

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**Theorem 1.** *Let $G$ be a group that is of bounded order. If $E$ is a subset of $G$ with*

$$\underline{\lim}\frac{N_n(E)}{\log[G:H_n]} < \infty, \tag{1}$$

*then $E$ is a set of uniqueness.*

*Proof*. Choose $\varepsilon > 0$ so that $\gamma \in \Gamma$ and $|\gamma(x) - 1| < \varepsilon$ implies $\gamma(x) = 1$. This is possible since $G$ is of bounded order. Without loss of generality, we will assume that $N_n(E) < C\log[G : H_n]$ for all $n \geq 0$. Now choose $r$ to be an integer such that $r > 2C\log(\pi/\sin^{-1}(\varepsilon/2))$. For each $n$, let $L_n$ be a finite set with $L_nH_n = EH_n$ and with $\operatorname{card}(L_n) < C\log[G : H_n]$. Break $L_n$ into $r$ disjoint subsets $L_{j,n}$ of approximately equal size. There are at most $Cr^{-1}\log[G:H_n]$ elements in each $L_{j,n}$. But

$$\left(\frac{\pi}{\sin^{-1}(\varepsilon/2)}\right)^{Cr^{-1}\log[G:H_n]} < [G:H_n]^{1/2} < [G:H_n]. \tag{2}$$

Since there are at most $\pi/\sin^{-1}(\varepsilon/2)$ elements of the unit circle such that all distances are more than $\varepsilon$, there are $\gamma_1, \gamma_2 \in H_n^{\perp}$ such that

$$|\gamma_1(x) - \gamma_2(x)| < \varepsilon \quad \text{for every } x \in L_{j,n}.$$

In fact, letting $\gamma = \gamma_2\gamma_1^{-1}$, we see that the cardianlity of the set

$$\Gamma_{j,n} = \{\gamma \in H_n^{\perp} : |\gamma(x) - 1| < \varepsilon \text{ for all } x \in L_{j,n}\}$$

is at least $[G : H_n]^{1/2}$ and so is unbounded for each $j$ as $n \to \infty$. If $(\gamma_1, \ldots, \gamma_p) \in \Gamma_{1,n} \times \cdots \times \Gamma_{r,n}$, then since each $\gamma_j \in H_n^{\perp}$ and $EH_n = L_nH_n$,

$$E \subseteq \bigcup_{j=1}^{r}\{x : |\gamma_j(x) - 1| < \varepsilon\}. \tag{3}$$

By our choice of $\varepsilon$, for such $(\gamma_1, \ldots, \gamma_p)$, we have

$$E \subseteq \bigcup_{j=1}^{r}\ker(\gamma_j). \tag{4}$$

For each $p \geq 1$, we choose particular $(\gamma_1, \ldots, \gamma_r)$ as follows. Choose $n$ large enough so that $\operatorname{card}\Gamma_{j,n} \geq (2^r - 1)\operatorname{card}H_p^{\perp}$.

Pick any $\gamma_1 \in \Gamma_{1,n}$. If $\gamma_1, \ldots, \gamma_{j-1}$ have been chosen, pick

$$\gamma_j \in \Gamma_{j,n}\backslash\bigcup\gamma_1^{\varepsilon_1}\gamma_2^{\varepsilon_2}\cdots\gamma_{j-1}^{\varepsilon_{j-1}}H_p^{\perp}$$

where the union is over all sequences $\varepsilon_1, \ldots, \varepsilon_{j-1}$ whose terms are all $-1$ or $0$. Because the cardinality of $\Gamma_{j,n}$ is large, such a choice is possible. The upshot is that no character of the form $\gamma_1^{\varepsilon_1}\cdots\gamma_r^{\varepsilon_r}$, where each $\varepsilon_j$ is 0 or 1, can be in $H_p^{\perp}$ unless each $\varepsilon_j = 0$. Now define

$$f_p = \prod_{j=1}^{r}(1 - \gamma_j).$$

Then $f_p = 0$ on the neighborhood $EH_n$ of $E$ and $(1-f_p)^\wedge(\chi) = 0$ if $\chi \in H_p^\perp$. If $S$ is any pseudofunction supported on $E$, we have $S(f_p) = 0$, so

$$|\widehat{S}(1)| \le (2^r - 1)\sup|\widehat{S}(\chi)|$$

where the supremum is taken over those $\chi \notin H_p^\perp$. Since $r$ is fixed and the supremum goes to 0 as $p \to \infty$, $\widehat{S}(1) = 0$. By a standard argument, $S = 0$. Thus $E$ is a U-set. □

It was previously known that if $N_n(E) = o(\log[G : H_n])$, then $E$ is a Dirichlet set and hence a strong U-set [2].

Now we turn to the question of the converse: how slow can $N_n(E)$ grow for an M-set? For the classical case, [3] shows that if $\varphi(\varepsilon) \to \infty$ as $\varepsilon \to 0$ there are sets in the circle group such that $N_\varepsilon(E) = O(\varphi(\varepsilon)\log(1/\varepsilon))$ and yet $E$ fails to be a U-set. To get this result, Kahane used the properties of Brownian motion. For the corresponding result in the current setting, we have to produce an analog of Brownian motion on Vilenkin groups. I would like to thank Robert Kaufman for his suggestions at this point.

Let $\xi_{xH_n}$ be a random point with values in $H_n$ which is uniformly distributed with respect to normalized Haar measure on $H_n$. We will assume that different $\xi_{xH_k}$ are independent random points. Thus, if $\gamma \in \Gamma$, we have

$$E[\gamma(\xi_{xH_n})] = \begin{cases} 1 & \text{if } \gamma \in H_n^\perp, \\ 0 & \text{otherwise}. \end{cases} \tag{5}$$

Notice that there is a separate random point for each coset of each $H_n$. We now define a random function on $G$ by

$$\psi(x) = \sum_{n=0}^{\infty} \xi_{xH_n}. \tag{6}$$

Notice that this sum converges for every point in $G$ and that for two points $x, y \in G$, if $xH_n = yH_n$ then $\psi(x)H_n = \psi(y)H_n$. Thus $\psi$ enjoys a type of Lipschitz continuity. Thus we have for a subset $E$ of $G$,

$$N_n(\psi(E)) \le N_n(E). \tag{7}$$

Now we present a converse to Theorem 1.

**Theorem 2.** *Let $G$ be a Vilenkin group. Assume that there is a constant $C$ such that*

$$\sum_{n=0}^{\infty} \frac{[G : H_{n+1}]}{[G : H_n]^C} < \infty. \tag{8}$$

*Let $\varphi(n) \uparrow \infty$ as $n \to \infty$. Then there exists an M-set $E$ in $G$ such that*

$$N_n(E) = O(\varphi(n)\log[G : H_n]). \tag{9}$$

*Proof.* Let $\theta(n) = \varphi(n)\log[G : H_n]$. Let $k_n$ be integers such that

$$\theta(n) \le k_1 k_2 \cdots k_n \le 2\theta(n).$$

Pick $k_n$ points of $H_n$ in distinct cosets of $H_{n+1}$ and let $\mu_n$ be a discrete measure on these points giving equal measure to each of them. Let $\mu = *_{n=1}^{\infty} \mu_n$. Then $\mu$ is a measure supported on a set $E$ with

$$N_n(E) \le 2\theta(n) \quad \text{and} \quad \mu(xH_n) \le 1/\theta(n) \quad \text{for all } n \ge 0.$$

Now consider the random measure $\nu$ where

$$\int f\,d\nu = \int f\circ\psi\,d\mu.$$

For $\gamma \in H_{n+1}^{\perp}\backslash H_n^{\perp}$, we have

$$\begin{aligned}\hat{\nu}(\gamma) &= \int_G \gamma\circ\psi\,d\mu = \sum_{xH_n}\int_{xH_n}\gamma\circ\psi\,d\mu = \sum_{xH_n}\int_{xH_n}\prod_{k=0}^{\infty}\gamma(\xi_{yH_k})\,d\mu(y)\\ &= \sum_{xH_n}\int_{xH_n}\prod_{k=0}^{n}\gamma(\xi_{xH_k})\,d\mu(y) = \sum_{xH_n}\mu(xH_n)\prod_{k=0}^{n}\gamma(\xi_{xH_k}).\end{aligned}\tag{10}$$

Notice that for $y \in xH_n$, it follows that $\xi_{xH_k} = \xi_{yH_k}$ for all $k \le n$. For each coset $xH_n$, we introduce the random variable

$$Y_{xH_n} = \mu(xH_n)\prod_{k=0}^{n}\gamma(\xi_{xH_k}).$$

Then $|Y_{xH_n}| = \mu(xH_n)$, and by taking conditional expectations with respect to all the $\xi_{xH_n}$ for this fixed $n$, we have that the $Y_{xH_n}$ are conditionally independent and

$$E[Y_{xH_n}|\{\xi_{xH_n} : x \in G\}] = 0.$$

Thus, using a sub-Gaussian inequality, we have for real $t$,

$$\begin{aligned}&E[\exp(t\,\mathrm{Re}\,\hat{\nu}(\gamma))|\{\xi_{xH_n} : x\in G\}]\\ &\quad= E\left[\exp\left(\sum_{xH_n} t\,\mathrm{Re}\,Y_{xH_n}\right)|\{\xi_{xH_n} : x \in G\}\right]\\ &\quad= \prod_{xH_n} E[\exp(t\,\mathrm{Re}\,Y_{xH_n})|\{\xi_{xH_n} : x\in G\}] \le \prod_{xH_n}\exp(t^2\mu(xH_n)^2)\\ &\quad= \exp\left(t^2\sum_{xH_n}\mu(xH_n)^2\right) \le \exp\left(\frac{t^2}{\theta(n)}\right).\end{aligned}\tag{11}$$

Now we have

$$E[\exp(t\,\mathrm{Re}\,\hat{\nu}(\gamma))] \le \exp\left(\frac{t^2}{\theta(n)}\right).\tag{12}$$

A similar expression holds for $\mathrm{Im}\,\hat{\nu}(\gamma)$. Recalling the definition of $\theta(n)$, we see

$$\begin{aligned}P[\mathrm{Re}\,\hat{\nu}(\gamma) > \varphi(n)^{-1/3}] &\le \exp\left(-\frac{1}{4}\varphi(n)^{1/3}\log[G : H_n]\right)\\ &\le \frac{1}{[G : H_n]^{\varphi(n)^{1/3}/4}}.\end{aligned}\tag{13}$$

This holds for each $\gamma \in H_{n+1}^{\perp}\backslash H_n^{\perp}$, so

$$P[\exists\gamma \in H_{n+1}^{\perp}\backslash H_n^{\perp} \text{ such that } \mathrm{Re}\,\hat{\nu}(\gamma) > \varphi(n)^{-1/3}] < \frac{[G : H_{n+1}]}{[G : H_n]^{\varphi(n)^{1/3}/4}}.\tag{14}$$

Since similar expressions hold for $-\operatorname{Re}\hat{\nu}(\gamma)$ and $\operatorname{Im}\hat{\nu}(\gamma)$ in place of $\operatorname{Re}\hat{\nu}(\gamma)$, and since $\varphi(n) \to \infty$, an application of the Borel-Cantelli Lemma along with (8) shows that $\lim \hat{\nu}(\gamma) = 0$ almost surely. □

Notice that all Vilenkin groups of bounded order satisfy the condition in (8). In fact, any group with growth like $\exp(\exp(n))$ satisfies this condition.

Using the methods above, we may prove the following result.

**Theorem 3.** *Let $G$ be a Vilenkin group satisfying the condition in* (8). *Given $\delta > 0$ there is a subset $E$ of $G$ such that $N_n(E) = O(\log[G : H_n])$ and a measure $\nu$ supported on $E$ with $\overline{\lim}|\hat{\nu}(\gamma)| \leq \delta$, where the limit is taken as $\gamma \to \infty$ in $\Gamma$. By Theorem* 2 *of* [2], *we see that $E$ is not a Dirichlet set.*

*Proof.* In the previous proof, simply let $\varphi(n)$ be a constant where $\varphi(n)^{1/3} > 4C$ and $\varphi(n)^{-1/3} < \delta$. □

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