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RATIOS OF REGULATORS IN EXTENSIONS OF NUMBER FIELDS

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ABSTRACT. Let L/K be an extension of number fields. Then

$$\operatorname{Reg}(L)/\operatorname{Reg}(K) > c_{[L:\mathbf{Q}]}(\log|D_L|)^m,$$

where Reg denotes the regulator, D_L is the absolute discriminant of L, and $c_{[L:\mathbf{Q}]} > 0$ depends only on the degree of L. The nonnegative integer $m = m(L/K)$ is positive if L/K does not belong to certain precisely defined infinite families of extensions, analogous to CM fields, along which $\operatorname{Reg}(L)/\operatorname{Reg}(K)$ is constant. This generalizes some inequalities due to Remak and Silverman, who assumed that K is the rational field $\mathbf{Q}$, and modifies those of Bergé-Martinet, who dealt with a general extension L/K but used its relative discriminant where we use the absolute one.

1. INTRODUCTION

Remak [R1] laid down the principle that a number field ought to have a large regulator if and only if it has a large discriminant. In one direction this follows from work of Landau [L, Sie], who proved that $\sqrt{|D_L|}(\log|D_L|)^{[L:\mathbf{Q}]-1}$ is an upper bound for $\operatorname{Reg}(L)$. To obtain an inequality in the opposite sense, Remak considered the field $\mathbf{Q}(E_L)$ generated by the units E_L of L. The geometry of numbers tells us that $\mathbf{Q}(E_L)$ can be generated by integral elements (units) whose size at every embedding is bounded in terms of $\operatorname{Reg}(L)$. It follows that $|D_{\mathbf{Q}(E_L)}|$ can be bounded above by a function of $\operatorname{Reg}(L)$. Remak then observed that $\mathbf{Q}(E_L) = L$ unless L is a CM field (a totally imaginary quadratic extension of a totally real field). Thus he proved [R1]

$$\operatorname{Reg}(L) > C_N \log\left(\frac{|D_L|}{N^N}\right), \tag{1.1}$$

where L is assumed non-CM, $N = [L:\mathbf{Q}]$, and $C_N > 0$ depends explicitly on N. In 1984 Silverman [Sil] improved the dependence on $\log|D_L|$ in (1.1) to

$$\operatorname{Reg}(L) > 2^{-4N^2}\left(\log\left(\frac{|D_L|}{N^{N^{\log_2(8N)}}}\right)\right)^{r_L-\rho},$$

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where $|D_L| > N^{N^{\log_2(8N)}}$ is assumed, r_L is the unit rank of L, and $\rho = \max_{F\subsetneq L}\{r_F\}$.

It follows from (1.1) that given an integer N and a real number y there are only finitely many non-CM number fields L such that $[L:\mathbf{Q}] \le N$ and $\mathrm{Reg}(L) < y$. CM fields must be excluded since the regulator, being essentially that of a proper subfield, can have the same value for infinitely many CM fields. We can, however, drop all restrictions on the degree $[L:\mathbf{Q}]$ by using Zimmert's [Z] bound

$$\mathrm{Reg}(L) > (0.04)1.05^{[L:\mathbf{Q}]}.$$

In the late 1980s Bergé and Martinet [BM1, BM2] generalized Remak and Silverman's method to the relative case. Given an extension L/K of number fields their idea was to equate the ratio of regulators $\mathrm{Reg}(L)/\mathrm{Reg}(K)$ with the covolume of a lattice produced from the units of L. In their approach the absolute norm $\mathrm{N}(\mathscr{D}_{L/K})$ of the relative discriminant of L/K appeared naturally and they were able to bound $\mathrm{Reg}(L)/\mathrm{Reg}(K)$ from below by a power of $\log(\mathrm{N}(\mathscr{E}_{L/K}))$.

While Bergé and Martinet's results can be used quite effectively [BM3] if $\mathrm{N}(\mathscr{D}_{L/K})$ is large, they are otherwise not so strong. This makes it difficult to obtain inequalities in which K is allowed to vary, say only fixing $[L:\mathbf{Q}]$, as there will be in general infinitely many L/K with $\mathrm{N}(\mathscr{D}_{L/K}) = 1$. Our results for totally real fields [CF] suggest that this problem could be overcome by modifying Bergé and Martinet's lattice. We use the lattice associated to the relative units $E_{L/K}$. By definition, $E_{L/K}$ consists of those units of L whose norm to K is a root of unity. Since the covolume of $E_{L/K}$ under the logarithmic embedding is readily related to $\mathrm{Reg}(L)/\mathrm{Reg}(K)$, we can apply Remak's geometric method to bound the absolute discriminant of $\mathbf{Q}(E_{L/K})$ from above in terms of $\mathrm{Reg}(L)/\mathrm{Reg}(K)$. It turns out that $\mathbf{Q}(E_{L/K}) = L$, except when one of the following three conditions holds:

(i) $L = K$.
(ii) The field L is CM (and K is any subfield of L).
(iii) There is a CM field M with maximal totally real subfield k such that K is a quadratic extension of k, $K \ne M$, and $L = MK$.

We call the extension L/K *unit-weak* if it satisfies (i), (ii) or (iii) above.

Theorem. *Let $E_{L/K}$ denote, as above, the relative units of an extension L/K of number fields. Assume that $|D_L| > 3N^N$, where D_L is the discriminant of $L/\mathbf{Q}$ and $N = [L:\mathbf{Q}]$. Then*

$$\frac{\mathrm{Reg}(L)}{\mathrm{Reg}(K)} > \frac{C}{N^{2r}}\left(\log\left(\frac{|D_L|}{N^N}\right)\right)^m, \tag{1.2}$$

where $r = \mathrm{rank}(E_{L/K}) = r_L - r_K$ is the difference of the unit ranks of L and K, Reg *is the regulator, and $C > 0$ is a computable absolute constant. The non-negative integer m is positive if L/K is not unit-weak* (*see the above definition*). *In general, $m = m(L/K) = r - \max_{F\subsetneq L}\{\mathrm{rank}(E_{L/K}\cap F)\}$, where the maximum is taken as F runs over all proper subfields of L.*

We actually prove the somewhat stronger inequality (3.7) in which $\mathrm{Reg}(L)/\mathrm{Reg}(K)$ is replaced by the regulator of $E_{L/K}$. The exponent m of $\log(|D_L|)$ in (1.2) is likely to be best possible. In any case, m can be computed

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by easy linear algebra, without knowledge of any unit, as long as one knows all the subfields of L (see the end of §3). In contrast, we do not calculate C here, as we do not obtain a good value. Our proof does yield that one can take $C = 1$ and $m = r$, provided we assume that every proper subfield of L is actually a subfield of K.

When L/K is unit-weak m vanishes and (1.2) becomes almost useless, however, in this case the ratio of regulators $\mathrm{Reg}(L)/\mathrm{Reg}(K)$ is essentially that of a proper subextension. Unit-weak extensions can thus be treated inductively and represent no essential complication to the problem of bounding $\mathrm{Reg}(L)/\mathrm{Reg}(K)$ from below. We treat unit-weak extensions briefly at the end of §§2 and 3.

A consequence of (1.2) is

Corollary. *Given an integer N and a real number y, there are at most finitely many extensions L/K such that $[L:\mathbf{Q}] \le N$, $\mathrm{Reg}(L)/\mathrm{Reg}(K) < y$, and L/K is not unit-weak.*

If L is totally real even more is true: Given any real number y there are finitely many pairs of *totally real* fields L and K, with $K \subsetneq L$, such that $\mathrm{Reg}(L)/\mathrm{Reg}(K) < y$ [CF]. We do not know yet if this extends to all non-unit-weak L/K, totally real or not.

2. The field generated by the relative units

Recall that the group of relative units $E_{L/K}$ of an extension L/K of number fields is defined by

$$E_{L/K} = \{\alpha \in E_L \mid \mathrm{Norm}_{L/K}(\alpha) \in W_K\},$$

where E_L denotes the units of L and W_K the torsion subgroup of E_K. The (free) rank of $E_{L/K}$ is $r = r_{L/K} = r_L - r_K$, where r_L is the rank of E_L. Let $\mathcal{S}_L$ denote the set of embeddings of L into $\mathbf{C}$. We embed E_L/W_L into $\mathbf{R}^{\mathcal{S}_L}$ by the map $\mathcal{L} = \mathcal{L}_L \colon E_L \to \mathbf{R}^{\mathcal{S}_L}$ defined by

$$(\mathcal{L}_L(\alpha))_\sigma = (\mathcal{L}(\alpha))_\sigma = \log|\sigma(\alpha)|, \qquad \sigma \in \mathcal{S}_L. \tag{2.1}$$

We endow $\mathbf{R}^{\mathcal{S}_L}$ with the Euclidean inner product

$$\langle (x_\sigma), (y_\sigma) \rangle = \sum_{\sigma \in \mathcal{S}_L} x_\sigma y_\sigma. \tag{2.2}$$

Then $\mathcal{L}_L(E_{L/K})$ is perpendicular to $\mathcal{L}_L(E_K)$. A dimension count shows that the $\mathbf{Q}$-spans $\mathbf{Q}\mathcal{L}_L(E_{L/K})$ and $\mathbf{Q}\mathcal{L}_L(E_K)$ of these two lattices are orthogonal complements of each other inside $\mathbf{Q}\mathcal{L}(E_L)$.

Our first goal is to characterize the extensions L/K for which $\mathbf{Q}(E_{L/K})$ is a proper subfield of L. Slightly more generally, we prove

Proposition 1. *Let L/K be an extension of number fields and let $E_{L/K}$ be its group of relative units. Let E be a subgroup of finite index in $E_{L/K}$ and suppose that E is contained in a proper subfield of L. Then at least one of* (i), (ii), *or* (iii) *below holds:*

(i) *$L = K$.*
(ii) *L is CM (and $K \subset L$ is arbitrary).*
(iii) *There is a CM field M with maximal totally real subfield k such that K is a quadratic extension of k, $K \ne M$, and $L = MK$.*

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Conversely, if (iii), (ii), *or* (i) *holds* (*with* $L \neq \mathbf{Q}$), *then* $E_{L/K}$ *contains a subgroup* E *as above.*

Proof. The last statement is obvious in cases (i) and (ii). If (iii) holds, let $H \neq K$, $H \neq M$ be the third field lying strictly between k and L. A short computation shows that $E := E_{H/k} \subset H$ has the same rank as $E_{L/K}$ and $E_{H/k} \subset E_{L/K}$, proving the converse claim.

We now prove the first part of the proposition. Given a subfield $F \subset L$ and an archimedean place ω of L, let $e_F(\omega) = e_{L/F}(\omega) = 2$ if ω ramifies in L/F; otherwise, let $e_F(\omega) = 1$. Let ∞_F denote the set of archimedean places of F. Then

$$r_F + 1 = \frac{1}{[L:F]} \sum_{\omega \in \infty_L} e_F(\omega), \tag{2.3}$$

because

$$r_F + 1 = \sum_{\nu \in \infty_F} 1 = \sum_{\nu \in \infty_F} \frac{1}{[L:F]} \sum_{\substack{\omega \in \infty_L \\ \omega | \nu}} e_F(\omega) = \frac{1}{[L:F]} \sum_{\omega \in \infty_L} e_F(\omega).$$

Let $H = \mathbf{Q}(E)$. Then $H \subsetneqq L$, by assumption. Since $E \subset E_H$, we have $r_H \geq r_{L/K} = r_L - r_K$. From this and (2.3) we obtain

$$\frac{1}{[L:H]} \sum_{\omega \in \infty_L} e_H(\omega) + \frac{1}{[L:K]} \sum_{\omega \in \infty_L} e_K(\omega) > \sum_{\omega \in \infty_L} 1.$$

The compositum $HK \subset L$ contains E and E_K. Modulo torsion, these are disjoint (perpendicular!) subgroups of E_L/W_L of rank $r_L - r_K$ and r_K; hence, the units of HK have rank r_L. If $HK \neq L$, then L must be a CM field, in which case the proof is done. We may therefore assume $HK = L$. Then we cannot simultaneously have $e_H(\omega) = 2$ and $e_K(\omega) = 2$ for $\omega \in \infty_L$. Hence,

$$\left(\frac{1}{[L:H]} + \frac{1}{[L:K]}\right) \sum_{\omega \in \infty_L} 1 + \max\left(\frac{1}{[L:H]}, \frac{1}{[L:K]}\right) \sum_{\omega \in \infty_L} 1 > \sum_{\omega \in \infty_L} 1. \tag{2.4}$$

By assumption, $[L:H] \geq 2$. Thus, either $[L:H] = 2$ or $[L:K] = 2$ (we dismiss the trivial case $L = K$).

We first assume $[L:K] = 2$. Let τ be the nontrivial element of $\mathrm{Gal}(L/K) \cong \mathbf{Z}/2\mathbf{Z}$. For $\alpha \in E \subset E_{L/K}$, we have $\mathrm{Norm}_{L/K}(\alpha) \in W_K$; therefore, $\tau(\alpha) = \eta\alpha^{-1}$, $\eta \in W_K$. By passing, as we may, to a subgroup of finite index in E, we can assume $\tau(\alpha) = \alpha^{-1}$; hence, τ induces a nontrivial field automorphism of $H = \mathbf{Q}(E)$. Let H_τ be its fixed field so that $[H:H_\tau] = 2$. Since $H_\tau \subset L_\tau = K$, we must have either $H \cap K = H_\tau$ or $H \cap K = H$. In the latter case we would have $E \subset K$. But then $E \subset K \cap E_{L/K} = W_K$. Since E has finite index in $E_{L/K}$, this could only happen if L is CM. We may thus assume $H \cap K = H_\tau$. Then $E \subset H \cap E_{L/K} = E_{H/H\cap K} \subset E_{L/K}$. Since E has finite index in $E_{L/K}$, $r_{H/H\cap K} = r_{L/K}$. From this and (2.3) we find

$$\frac{1}{[L:H]} \sum_{\omega \in \infty_L} e_H(\omega) - \frac{1}{[L:H\cap K]} \sum_{\omega \in \infty_L} e_{H\cap K}(\omega) = \sum_{\omega \in \infty_L} 1 - \frac{1}{2} \sum_{\omega \in \infty_L} e_K(\omega).$$

Since $[L:H\cap K] = 2[L:H]$, we have

$$\frac{1}{[L:H]} \sum_{\omega \in \infty_L} (2e_H(\omega) - e_{H\cap K}(\omega)) = \sum_{\omega \in \infty_L} (2 - e_K(\omega)). \tag{2.5}$$

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Observe that if ω ramifies in L/K, then ω ramifies in $L/H \cap K$ but not in L/H (since $L = HK$). Thus, if $e_K(\omega) = 2$, then $2e_H(\omega) - e_{H\cap K}(\omega) = 0$. If $e_K(\omega) = 1$, then $2e_H(\omega) - e_{H\cap K}(\omega) \leq 2$. It now follows from (2.5) that $[L : H] = 2$ and that $e_H(\omega) = 2$ if and only if $e_K(\omega) = 1$. Hence $[L : H] = 2 = [H : H \cap K] = [K : H \cap K]$ and all archimedean places of L ramify in either L/K or L/H, but none in both extensions. It follows that L/K satisfies condition (iii) in the proposition (let $k = K \cap H$ and let $M \neq K$, $M \neq H$, be the third field lying strictly between k and L). This proves Proposition 1 when $[L : K] = 2$.

If $[L : K] > 2$, then (2.4) implies $[L : H] = 2$. The strategy now is to reverse the roles of H and K and thereby reduce the proof to the quadratic case that we just handled. Recall that if F is any subfield of L, then the **Q**-spans of $\mathscr{L}(E_{L/F})$ and $\mathscr{L}(E_F)$ are orthogonal with respect to the (**R**-valued) inner product (2.2). By construction, $\mathscr{L}(E) \subset \mathscr{L}(E_H)$. Since E has finite index in $E_{L/K}$, $\mathbf{Q}\mathscr{L}(E) = \mathbf{Q}\mathscr{L}(E_{L/K})$. Hence

$$\mathbf{Q}\mathscr{L}(E_{L/H}) = \mathbf{Q}\mathscr{L}(E_H)^{\perp} \subset \mathbf{Q}\mathscr{L}(E_{L/K})^{\perp} = \mathbf{Q}\mathscr{L}(E_K), \tag{2.6}$$

where $^{\perp}$ denotes the orthogonal complement inside $\mathbf{Q}\mathscr{L}(E_L)$. Since the kernel W_L of $\mathscr{L}$ is finite, (2.6) shows that $E^n_{L/H} \subset E_K$ for some positive integer n. Thus $E' := E^n_{L/H}$ has finite index in $E_{L/H}$, $[L : H] = 2$, and $\mathbf{Q}(E') \subset K$, a proper subfield of L; but this is the quadratic case of the proposition, so the proof is done.

We conclude this section with a brief discussion of the unit-index $u_{L/K}$ of a unit-weak extension L/K. We assume first that $K \neq L$ and that L is not CM. Let k and M be as in (iii) above. Denote by K and H the two remaining fields lying strictly between k and L. Let τ_H, τ_K, and $\tau_M = \tau_H\tau_K$ be the nontrivial automorphisms of L/H, L/K, and L/M. Since we assume that L is not CM, at least one archimedean place of k ramifies in H; hence, at least one archimedean place of K ramifies in L. Thus $W_K = \{\pm 1\}$ and -1 is not a norm in L/K, whence $\mathrm{Norm}_{L/K}(E_{L/K}) = \{+1\}$. Equivalently, $\tau_K(\alpha) = \alpha^{-1}$ for $\alpha \in E_{L/K}$. Hence, $\mathrm{Norm}_{L/M}(\alpha) = \alpha\tau_H(\tau_K(\alpha)) = \alpha/\tau_H(\alpha)$. Therefore, $\mathrm{Norm}_{L/M}(\alpha) = 1$ if and only if $\alpha \in E_{L/K} \cap H = E_{H/k}$. In short, $\mathrm{Norm}_{L/M}$ induces an injection of $E_{L/K}/E_{H/k}$ into $W_M = E_{M/k}$. As $W_M^2 = \mathrm{Norm}_{L/M}(W_m) \subset \mathrm{Norm}_{L/M}(W_L)$ and W_M is cyclic, we have $u_{L/K} := [E_{L/K} : W_L E_{H/k}] = 1$ or 2.

So far we have assumed that L is not CM. If L is CM, let H be its maximal totally real subfield. It is well known that $[E_L : W_L E_H] = 1$ or 2 [R2]. It follows that $u_{L/K} := [E_{L/K} : W_L E_{H/k}] = 1$ or 2, where $k = H \cap K$. Finally, if $L = K$ we let $H = k = \mathbf{Q}$ and $u_{L/K} = 1$.

We have thus defined, whenever L/K is unit-weak, a subextension H/k and a unit-index $u_{L/K} := [E_{L/K} : W_L E_{H/k}] = 1$ or 2. When L is CM and $K = \mathbf{Q}$, $u_{L/\mathbf{Q}}$ is just the usual unit-index of L. In the next section we relate the regulators of $E_{L/K}$ and $E_{H/k}$ using $u_{L/K}$. Notice that H/k itself is not unit-weak unless $r_{L/K} = 0$.

3. Proof of Theorem

We begin with the definition of the regulator of relative units $\mathrm{Reg}(E_{L/K})$. Pick $\alpha_1, \alpha_2, \ldots, \alpha_r$ to be independent generators of $E_{L/K}/W_L$, the relative

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units modulo torsion. Let M be the matrix $M = (\log\|\alpha_i\|_\omega)$, where $1 \le i \le r$, ω runs over the set ∞_L of archimedean places of L, and $\|\ \|_\omega$ denotes the normalized absolute value at ω (so that $\|\ \|_\omega = |\ |_\omega^2$ if ω is complex, and $\|\ \|_\omega = |\ |_\omega$ otherwise). For each place $\nu \in \infty_k$, fix a place $\omega_\nu \in \infty_L$ lying above ν. Then $\mathrm{Reg}(E_{L/K})$ is the absolute value of the determinant of the submatrix of M, which results when we delete from M the rows corresponding to the ω_ν's. In [CF, Theorem 1] we showed, for L/K of any signature,

$$\mathrm{Reg}(E_{L/K}) = \frac{1}{[E_K : W_K \,\mathrm{Norm}_{L/K}(E_L)]} \frac{\mathrm{Reg}(L)}{\mathrm{Reg}(K)}. \tag{3.1}$$

We also related [CF, Lemma 2.1] $\mathrm{Reg}(E_{L/K})$ to the r-dimensional volume $V_L(E_{L/K})$ of a fundamental domain for $\mathscr{L}(E_{L/K})$ (see (2.1)),

$$V_L(E_{L/K}) = [L:K]^{(r_1(K)+r_2(K))/2} 2^{(r_2(K)-r_2(L))/2} \,\mathrm{Reg}(E_{L/K}), \tag{3.2}$$

where (r_1, r_2) denotes the number of (real, complex) places. The Euclidean structure (which normalizes volume) is given by $\|(x_\sigma)\|^2 = \langle (x_\sigma), (x_\sigma)\rangle$, as in (2.2). For $\alpha \in E_L$ we write $\|\alpha\|$ instead of $\|\mathscr{L}(\alpha)\|$. Thus,

$$\|\alpha\|^2 := \sum_{\sigma\in\mathscr{S}_L} (\log|\sigma(\alpha)|)^2, \tag{3.3}$$

where $\mathscr{S}_L$ denotes the set of all embeddings of L into $\mathbf{C}$. We will need the lower bound [F, (3.21)]

$$\|\alpha\| > \frac{C'}{\sqrt{N}(\log N)^3}, \tag{3.4}$$

where $\alpha \in E_L$, $\alpha \notin W_L$, $N = [L:\mathbf{Q}]$, and $C' > 0$ is a computable absolute constant (inequality (3.4) follows easily from Dobrowolsky's lower bound for heights [D]).

Let the successive minima of $\|\ \|$ on the lattice $\mathscr{L}(E_{L/K})$ be attained at $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r$. Thus [GK, pp. 195, 197] the subgroup $E := \langle\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r\rangle$ of $E_{L/K}$ generated by the ε_i has finite index in $E_{L/K}$ and

$$0 < \|\varepsilon_1\| \le \|\varepsilon_2\| \le \cdots \le \|\varepsilon_r\|, \tag{3.5}$$

$$\prod_{i=1}^{r} \|\varepsilon_i\| \le \gamma_r^{r/2} V_L(E_{L/K}), \tag{3.6}$$

where γ_r denotes Hermite's constant in dimension $r = r_{L/K}$.

Lemma. *Let $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r$ be as above and assume that L/K is not unit-weak (see* §1*). Let $H_0 = \mathbf{Q}$ and $H_i = H_{i-1}(\varepsilon)$. Then there is an integer T such that $H_T \ne L$, $H_{T+1} = L$, $0 \le T < r$, and*

$$\frac{1}{[L:\mathbf{Q}]}\log|D_L| \le \log([L:\mathbf{Q}]) + \frac{1}{\sqrt{3[L:\mathbf{Q}]}}\sum_{i=1}^{T+1}\|\varepsilon_i\|\sqrt{[H_i:H_{i-1}]^2-1},$$

where D_L denotes the absolute discriminant of L and $\|\ \|$ is given by (3.3).

Proof. Proposition 1 implies that there is at least $T < r$ so that $L = \mathbf{Q}(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{T+1})$. The inequality then follows from [F, (3.3), (3.14) and Lemma 3.5].

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Theorem. *Let L/K be an extension of number fields and assume that $D_L > 3N^N$, where D_L is the absolute discriminant of L and $N = [L : \mathbf{Q}]$. Then*

$$\operatorname{Reg}(E_{L/K}) \geq \frac{C}{N^{2r}} \left(\log \left(\frac{|D_L|}{N^N} \right) \right)^m . \tag{3.7}$$

Here $\operatorname{Reg}(E_{L/K})$ is the regulator of relative units given by (3.1), $C > 0$ is a computable absolute constant, and $r = r_L - r_K = \operatorname{rank}(E_{L/K})$ is the difference of the unit ranks of L and K. The nonnegative integer m is positive if L/K is not unit-weak (see §1). In general, $m = m(L/K) = r - \max_{F \subsetneq L}\{\operatorname{rank}(E_{L/K} \cap F)\}$, where F runs over all proper subfields of L.

The slightly simplified version of the theorem given in §1 follows from (3.1) and (3.7).

Proof. We first assume that L/K is not unit-weak. From the Lemma and (3.5) we have

$$\frac{1}{N} \log \left(\frac{|D_L|}{N^N} \right) \leq \frac{\|\varepsilon_{T+1}\|}{\sqrt{3N}} \sum_{i=1}^{T+1} \sqrt{[H_i : H_{i-1}]^2 - 1} \leq \|\varepsilon_{T+1}\| \sqrt{\frac{N}{3}}, \tag{3.8}$$

since $\prod_{i=1}^{T+1} [H_i : H_{i-1}] = N$. From (3.5), (3.6), and (3.4)

$$\|\varepsilon_{T+1}\|^{r-T} \leq \prod_{i=T+1}^{r} \|\varepsilon_i\| \leq \frac{\gamma_r^{r/2} V_L(E_{L/K})}{(C'/\sqrt{N}(\log N)^3)^T} . \tag{3.9}$$

If we put this together with (3.2) and (3.8) and use $\log(|D_L|/N^N) > 0$, we find

$$\begin{aligned} &\frac{1}{N^{2r}} \left(\log \left(\frac{|D_L|}{N^N} \right) \right)^{r-T} \\ &\leq \frac{(([L : K]/2)^{(r_1(K)+r_2(K))/r} 2^{([K : \mathbf{Q}] - r_2(L))/r} \gamma_r / 3N)^{r/2}}{(NC'/\sqrt{3}(\log N)^3)^T} \operatorname{Reg}(E_{L/K}) . \end{aligned} \tag{3.10}$$

If $[L : K] \geq 3$, then (2.3) yields

$$r = r_L - r_K = \sum_{\omega \in \infty_L} \left(1 - \frac{e_K(\omega)}{[L : K]} \right) \geq \sum_{\omega \in \infty_L} \frac{1}{3} \geq \frac{[L : \mathbf{Q}]}{6} .$$

Hence, for $[L : K] \geq 2$,

$$\left(\frac{[L : K]}{2} \right)^{(r_1(K)+r_2(K))/r} \leq \left(\frac{[L : K]}{2} \right)^{6/[L : K]} < 3.003. \tag{3.11}$$

Note that

$$[K : \mathbf{Q}] - r_2(L) \leq r_1(L) + r_2(L) - r_1(K) - r_2(K) = r \tag{3.12}$$

and that, for $r > 2$, $\gamma_r \leq r/2.1$. (*Proof.* Use the inequalities quoted in [CF, (2.9)]). We then have in (3.10)

$$\left(\left(\frac{[L : K]}{2} \right)^{(r_1(K)+r_2(K))/r} 2^{([K : \mathbf{Q}] - r_2(L))/r} \frac{\gamma_r}{3N} \right)^{r/2} \leq 1 , \tag{3.13}$$

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for all $r > 0$ (do $e = 1$ or 2 separately). Since $T < r < N$, (3.10) and (3.13) yield

$$\text{Reg}(E_{L/K}) > \frac{C}{N^{2r}} \left(\log \left(\frac{|D_L|}{N^N} \right) \right)^{r-T}, \tag{3.14}$$

with $C > 0$ a computable absolute constant. To prove (3.7) we must still show that in (3.14) we can replace T by $\rho := \max_{F \subsetneq L} \{\text{rank}(E_{L/K} \cap F)\}$. Since we assume $D_L > 3N^N$, it suffices to show $T \leq \rho$. By the lemma, H_T is a proper subfield of L containing the T independent relative units $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_T \in E_{L/K}$; hence, $T \leq \rho$. Proposition 1 implies that $m = r - \rho > 0$ which concludes the proof when L/k is not unit-weak.

If L/K is unit-weak then $m = r - \rho = 0$ in (3.7). In this case (3.7) follows from

Proposition 2. *Let L/K be an extension of number fields. Then*

$$\text{Reg}(E_{L/K}) \geq \frac{c^r}{(Nr(\log N)^6)^{r/2}}. \tag{3.15}$$

Here $\text{Reg}(E_{L/K})$ *is the regulator of relative units given by* (3.1), $c > 0$ *is a computable absolute constant,* $N = [L : \mathbf{Q}]$, *and* $r = r_L - r_K$ *is the difference of the unit ranks of L and K.* (*If* $r = 0$, (3.15) *means the trivial* $1 \geq 1$.)

Proof. From (3.4), (3.6), and (3.2) we obtain

$$\text{Reg}(E_{L/K}) \geq \left(\frac{C'^2}{N\gamma_r(\log N)^6([L:K]/2)^{(r_1(K)+r_2(K))/r} 2^{([K:\mathbf{Q}]-r_2(L))/r}} \right)^{r/2}.$$

Now use (3.11), (3.12), and $\gamma_r \leq r$ to obtain (3.15), with $c = C'\sqrt{6.006}$.

Corollary. *Let L/K (and all notation) be as in the theorem. Suppose further that all proper subfields of L are actually subfields of K. Then*

$$\text{Reg}(E_{L/K}) \geq \frac{1}{N^{2r}} \left(\log \left(\frac{|D_L|}{N^N} \right) \right)^r. \tag{3.16}$$

Proof. We first dispose of the trivial cases. If L/K is unit-weak, the hypothesis on K implies that case (iii) in Proposition 1 cannot hold. If (ii) holds, so L is CM, then K must be its maximal totally real subfield. Then $r = 0$ and (3.16) is trivial. Since case (i) ($L = K$) is equally trivial, we may assume that L/K is not unit weak. Consider, in the notation of the lemma, $H_1 = \mathbf{Q}(\varepsilon_1)$. By assumption, either $H_1 \subseteq K$ or $H_1 = L$. But $H_1 \subseteq K$ implies $\varepsilon_1 \in E_{L/K} \cap K$, which is impossible since ε_1 is not a root of unity. Thus, $H_1 = L$ and so $T = 0$ in the lemma. The corollary now follows from (3.13) and (3.10).

The computation of $m = m(L/K)$ in the theorem turns out to be elementary. Let $\mathscr{L}_L$ be the logarithmic embedding (2.1). If $M \subset \mathscr{L}_L(E_L) \subset \mathbf{R}^{\mathscr{S}_L}$ is a lattice, denote its $\mathbf{R}$-span by $\mathbf{R}M$. Thus, $\text{rank}(M) = \dim_{\mathbf{R}}(\mathbf{R}M)$. If F is a subfield of L, observe that

$$\begin{aligned}
&\text{rank}(E_{L/K}) + \text{rank}(E_F) - \text{rank}(E_{L/K} \cap F) \\
&= r_L - r_K + r_F - \text{rank}(E_{L/K} \cap E_F) \\
&= \text{rank}(E_{L/K}E_F) = \dim_{\mathbf{R}}(\mathbf{R}\mathscr{L}_L(E_{L/K}E_F)) \\
&= \dim_{\mathbf{R}}(\mathbf{R}\mathscr{L}_L(E_{L/K}) + \mathbf{R}\mathscr{L}_L(E_F)).
\end{aligned}$$

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Dirichlet's unit theorem gives an $\mathbf{R}$-basis of $\mathbf{R}\mathscr{L}_L(E_F)$. It also gives one for $\mathbf{R}\mathscr{L}_L(E_{L/K})$ as the orthogonal complement of $\mathbf{R}\mathscr{L}_L(E_K)$ (inside $\mathbf{R}\mathscr{L}_L(E_L)$). It follows that

$$m := \operatorname{rank}(E_{L/K}) - \max_{F \subsetneq L}\{\operatorname{rank}(E_{L/K} \cap F)\}$$

can be calculated by linear algebra from a knowledge of all the subfields of L, without knowing a single unit. To be precise, one has to know, for each subfield F of L, the mapping $\mathscr{S}_L \to \mathscr{S}_F$ obtained by restricting the embeddings of L to embeddings of F.

We conclude with a comment on $\operatorname{Reg}(E_{L/K})$ and $\operatorname{Reg}(L)/\operatorname{Reg}(K)$ for L/K unit-weak. We defined in §2 a subextension H/k and a unit index

$$u_{L/K} := [E_{L/K} : W_L E_{H/k}] = 1 \text{ or } 2.$$

On examining the ramification of the archimedean places in L/K and H/k one finds, directly from the definition of $\operatorname{Reg}(E_{L/K})$ as a determinant,

$$\operatorname{Reg}(E_{L/K}) = 2^{r_{H/k}} \operatorname{Reg}(E_{H/k})/u_{L/K}. \tag{3.17}$$

If we let L/K range over the infinitely many unit-weak extensions associated to the same H/k, it is clear from (3.17) that $\operatorname{Reg}(E_{L/K})$ assumes at most two values. It follows, mainly from (3.1), that $\operatorname{Reg}(L)/\operatorname{Reg}(K)$ assumes at most $2^{[H:\mathbf{Q}]}$ values.

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