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CONJUGATION AND EXCESS IN THE STEENROD ALGEBRA

JUDITH H. SILVERMAN

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ABSTRACT. In this paper we prove a formula involving the canonical antiautomorphism χ of the mod-2 Steenrod algebra $\mathscr{A}(2)$, namely,

$$\chi(Sq^{2^j(2^{i+1}-1)}Sq^{2^{j-1}(2^{i+1}-1)}\cdots Sq^{(2^{i+1}-1)}) = Sq^{2^i(2^{j+1}-1)}Sq^{2^{i-1}(2^{j+1}-1)}\cdots Sq^{(2^{j+1}-1)},$$

and discuss its implications for the study of the image of the $\mathscr{A}(2)$-action on $\mathbb{F}_2[x_1, \ldots, x_s]$.

1. INTRODUCTION

1.1. **Notation.** The Milnor basis of the mod-2 Steenrod algebra is indexed by sequences $R = (r_1, r_2, \ldots)$ of nonnegative integers almost all of which are 0 [M]. We denote the corresponding basis element by $\langle R\rangle$; its dimension is $|\langle R\rangle| = \sum_i (2^i - 1)r_i$. If $R = (r, 0, 0, \ldots)$, then the corresponding basis element, abbreviated $\langle r\rangle$, is the Steenrod square Sq^r. The element $\langle a_1\rangle \cdots \langle a_n\rangle$ is *admissible* if $a_r \geq 2a_{r+1}$ for $r < n$ and $a_n > 0$ if $n > 1$. The admissible elements form an additive basis of $\mathscr{A}(2)$ [SE], but though the objects of study in this paper are certain admissible elements, the calculations are best expressed in terms of the Milnor basis. In particular, define $\mathscr{B}(\theta)$ for $\theta \in \mathscr{A}(2)$ to be the set of Milnor basis elements appearing in θ, so that $\theta = \sum\{\langle R\rangle : \langle R\rangle \in \mathscr{B}(\theta)\}$.

The Steenrod algebra acts on $\mathbb{F}_2[x_1, \ldots, x_s]$, the polynomial algebra on elements x_i of dimension 1, which is the mod-2 cohomology ring of

$$\overbrace{\mathbb{R}P^\infty \times \cdots \times \mathbb{R}P^\infty}^{s}.$$

Following Singer [Si], we say that a polynomial F is *hit* it it is the image of the positively graded part of $\mathscr{A}(2)$, that is, if $F = \sum_{i>0} Sq^i F_i$ for some polynomials F_i [Pe, W]. The *excess* of an element $\theta \in \mathscr{A}(2)$ is given by $\mathrm{ex}(\theta) = \min\{\hat{s} : \theta(x_1 x_2 \cdots x_{\hat{s}}) \neq 0 \in \mathbb{F}_2[x_1, \ldots, x_{\hat{s}}]\}$ (cf. [K]). Since linear maps commute with the action of $\mathscr{A}(2)$, it follows that whatever s might be, $\theta(F) = 0$ for any polynomial in $\mathbb{F}_2[x_1, \ldots, x_s]$ of degree $< \mathrm{ex}(\theta)$. One has that $\mathrm{ex}(\langle n\rangle) = n$

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and, more generally, that $\operatorname{ex}(\langle a_1\rangle\cdots\langle a_n\rangle)=\sum_{r=1}^{n-1}(a_r-2a_{r+1})+a_n$ if $\langle a_1\rangle\cdots\langle a_n\rangle$ is admissible [SE].

Given a pair (k, a) of nonnegative integers, we define $T(k; a)=\langle 2^k a\rangle\cdots\langle 2a\rangle\langle a\rangle$. Note that $|T(k; a)|=(2^{k+1}-1)a$, that $T(k; a)$ is admissible of excess a, and that on polynomials of degree $|A|=a$ one has $T(k; a)(A)=A^{2^{k+1}}$.

We denote by χ the canonical antiautomorphism of $\mathscr{A}(2)$.

1.2. **Results.** In [D] Davis computes $\chi T(j; 1)$:

Theorem 1.1 [D]. *For all* $j\geq 0$, *we have* $\chi T(j; 1)=\langle 2^{j+1}-1\rangle$.

Our Theorem 3.1 is a generalization of Theorem 1.1.

Theorem 3.1. *For all integers* i *and* j, *we have* $\chi T(j; 2^{i+1}-1)=T(i; 2^{j+1}-1)$.

Davis's argument proves also that $\chi T(j; 2)=\langle 2(2^{j+1}-1)\rangle$, leading one to conjecture the following.

Conjecture 1.2. $\chi T(j; 2(2^{i+1}-1))=T(i; 2(2^{j+1}-1))$ *for all* $j\geq 0$.

Bruner, using a computer, has verified that this conjecture holds for pairs i, j such that $i\leq 4$ and $|T(i; 2(2^{j+1}-1))|\leq 255$.

The pattern of these formulas for $\chi T(j; 2^k)$, $k=0, 1$, breaks down for $k=2$; it is not true that $\chi T(1; 4)=\langle 12\rangle$.

Theorem 3.1 and Conjecture 1.2, if it is true, allow us to identify new families of hit monomials, as we discuss in §4.

2. Technical lemma

We begin by recalling a proposition from the proof of Theorem 1.1. For integers $a=\sum a_i 2^i$ and $b=\sum b_i 2^i$, where a_i, $b_i\in\{0, 1\}$, we say *a dominates b* if $a_i\geq b_i$ for all i, and write $a\succeq b$.

Proposition 2.1 [D]. $\langle m\rangle\chi\langle n\rangle=\sum\{\langle R\rangle : |R|=m+n;\ |R|+\sum r_i\succeq 2m\}$.

An argument similar to Davis's describes the result when the order of factors is reversed.

Proposition 2.2. $\chi\langle n\rangle\cdot\langle m\rangle=\sum\{\langle R\rangle : |R|=m+n;\ \sum r_i\succeq m\}$.

Note that in both cases, the question of whether an $\langle R\rangle$ of the relevant degree appears as a summand depends only on $\sum r_i$. The formulas are very similar, and the following lemma, used repeatedly in the inductive proof of Theorem 3.1, combines them to give a way of pushing a factor involving χ through a product. Specifically, it permits us, in certain cases, to write $\langle m\rangle\chi\langle n\rangle$ as the sum of $\chi\langle p\rangle\cdot\langle q\rangle$ and two terms of the form $\langle m'\rangle\chi\langle n'\rangle$, which with luck will be more tractable than the original $\langle m\rangle\chi\langle n\rangle$.

Lemma 2.3. *Suppose that* k, l, m, *and* n *are nonnegative integers such that*:

(1) $k>l$,
(2) $m+n=2^k-2^l$,
(3) $m<2^{k-1}$,
(4) $m\equiv 0 \pmod{2^l}$.

Then

$$Sq(m)\cdot\chi Sq(n) = \chi Sq(n-m-2^l)\cdot Sq(2m+2^l) + Sq(m+2^{k-1}+2^{l-1}) \cdot\chi Sq(n-2^{k-1}-2^{l-1}) + Sq\left(\frac{m+n}{2}\right)\cdot\chi Sq\left(\frac{m+n}{2}\right),$$

with the understanding that $\chi Sq(s) = 0$ *if* $s < 0$, *and that the rightmost two terms vanish when* $l = 0$.

Proof. For simplicity, we will refer to the ith term in the equation above as Q_i. To prove the lemma, it is enough to show that $\mathscr{B}(Q_1) = \mathscr{B}(Q_2)\coprod\mathscr{B}(Q_3)\coprod\mathscr{B}(Q_4)$, where $X\coprod Y$ is the symmetric difference of the two sets. For $Sq(T) = Sq(t_1, t_2, \dots)$ a basis element of degree $|Sq(T)| = 2^kK - 2^l$, write $L_T = 2^kK - 2^l + \sum t_i$ and $R_T = \sum t_i$. By Propositions 2.1 and 2.2, $Sq(T)\in\mathscr{B}(Q_1)$ (resp. $\mathscr{B}(Q_2)$, resp. $\mathscr{B}(Q_3)$, resp. $\mathscr{B}(Q_4)$) $\Leftrightarrow L_T\succeq 2m$ (resp. $R_T\succeq 2m+2^l$, resp. $L_T\succeq 2^k+2m+2^l$, resp. $L_t\succeq 2^k-2^l$). By (3) and (4) above, $\mathscr{B}(Q_3)$ and $\mathscr{B}(Q_4)$ are $\subseteq\mathscr{B}(Q_1)$. Writing $L_T = (R_T - 2^l) + 2^kK$ and using (3) and (4), one finds that also $\mathscr{B}(Q_2)\subseteq\mathscr{B}(Q_1)$. Similarly, expressing R_T in terms of L_T, one may check that $\mathscr{B}(Q_1)\subseteq\mathscr{B}(Q_2)\coprod\mathscr{B}(Q_3)\coprod\mathscr{B}(Q_4)$. Observe that if $Sq(T)\in\mathscr{B}(Q_1)$, then $Sq(T)\in\mathscr{B}(Q_4)\Leftrightarrow\sum t_i < 2^l$. Finally, note that $\mathscr{B}(Q_1) = B(Q_2)$ when $l = 0$. This proves the lemma. □

3. Proof of the theorem

We are now ready to prove the main result of this paper.

Theorem 3.1. *For all integers* i *and* j, *we have* $\chi T(j; 2^{i+1}-1) = T(i; 2^{j+1}-1)$.

Proof. Clearly it suffices to prove the theorem under the assumption that $i\le j$. Theorem 2.1 gives the result for $i = 0$ and all j, so we proceed inductively by assuming that $\chi T(\hat\imath; 2^{\hat\jmath+1}-1) = T(\hat\jmath; 2^{\hat\imath+1}-1)$ for $\hat\imath\le i-1$ and all $\hat\jmath$ and for $\hat\imath = i$ and all $\hat\jmath\le j-1$. The inductive proof will draw on the following remark: under the above assumptions,

$$\chi Sq(2^{l-1}K)\cdot T(l-1; 2^{i+1}-1) = 0 \quad \text{for all } K\equiv 1 \pmod 2 \text{ and all } 1\le l\le i. \tag{1}$$

For we have

$$\begin{aligned}\chi Sq(2^{l-1}K)\cdot T(l-1; 2^{i+1}-1) &= \chi[\chi T(l-1; 2^{i+1}-1)\cdot Sq(2^{l-1}K)]\\ &= \chi[T(i; 2^l-1)\cdot Sq(2^{l-1}K)]\\ &= \chi[T(i-1; 2(2^l-1))Sq(2^l-1)\cdot Sq(2^{l-1}K)],\end{aligned}$$

and it follows from the Adem relations that $Sq(2^l-1)Sq(2^{l-1}K) = 0$. This verifies (1).

Now consider

$$\begin{aligned}\chi T(i; 2^{j+1}-1) &= \chi[T(i-1; 2^{j+1}-1)]\cdot\chi\langle 2^i(2^{j+1}-1)\rangle\\ &\overset{\text{ind}}{=} T(j; 2^i-1)\cdot\chi\langle 2^i(2^{j+1}-1)\rangle.\end{aligned}\tag{2}$$

Beginning from the right, we multiply $\chi\langle 2^i(2^{j+1}-1)\rangle$ by the successive terms $\langle 2^l(2^i-1)\rangle$, $0\le l\le j$, of $T(j; 2^i-1)$. At the cost of an error term, Lemma 2.3 allows us to push the factor involving χ past the lth singleton, transforming both in the process. Furthermore, (1) ensures that the error term vanishes upon multiplication by the $(l-1)$-singletons to its right. Namely, we prove that

$$\begin{aligned} T(l;\, 2^i-1)\cdot\chi\langle 2^i(2^{j+1}-1)\rangle \\ = \chi\langle 2^i(2^{j+1}-2^{l+1})\rangle\cdot T(l;\, 2^{i+1}-1) \quad \text{for } 0\le l\le j. \end{aligned} \tag{3}$$

The case $l=0$ follows from Lemma 2.3. Suppose then that the claim holds for $l-1$. Then, as $T(l;\, 2^i-1) = \langle 2^l(2^i-1)\rangle\cdot T(l-1;\, 2^i-1)$, we have

$$\begin{aligned} &T(l;\, 2^i-1)\cdot\chi Sq(2^i(2^{j+1}-1)) \\ &\quad = Sq(2^l(2^i-1))\cdot\chi Sq(2^i(2^{j+1}-2^l))\cdot T(l-1;\, 2^{i+1}-1) \quad \text{[induction]} \\ &\quad = [\chi Sq(2^i(2^{j+1}-2^{l+1}))\cdot Sq(2^l(2^{i+1}-1))]\cdot T(l-1;\, 2^{i+1}-1) \quad \text{[Lemma 2.3]} \\ &\qquad + [Sq(2^{i+j}+2^{i+l}-2^{l-1})\cdot\chi Sq(2^{i+j}-2^{i+l}-2^{l-1})]\cdot T(l-1;\, 2^{i+1}-1) \\ &\qquad + [Sq(2^{i+j}-2^{l-1})\cdot\chi Sq(2^{i+j}-2^{l-1})]\cdot T(l-1;\, 2^{i+1}-1). \end{aligned}$$

But the second and third summands vanish by (1), so we have

$$\begin{aligned} &T(l;\, 2^i-1)\cdot\chi Sq(2^i(2^{j+1}-1)) \\ &\quad = \chi Sq(2^i(2^{j+1}-2^{l-1}))\cdot[Sq(2^l(2^{i+1}-1))T(l-1;\, 2^{i+1}-1)] \\ &\quad = \chi Sq(2^i(2^{j+1}-2^{l-1}))\cdot T(l;\, 2^{i+1}-1). \end{aligned}$$

This establishes (3). Finally, taking $l=j$, we find that

$$\begin{aligned} \chi T(i;\, 2^{j+1}-1) &= T(j;\, 2^i-1)\cdot\chi\langle 2^i(2^{j+1}-1)\rangle \\ &= \chi\langle 0\rangle\cdot T(j;\, 2^{i+1}-1). \end{aligned}$$

This proves the theorem. □

4. Hit elements

In [K] Kraines gives a proof that the excess $\mathrm{ex}(\chi\langle n\rangle)$ is given by $\mu(n)$, where $\mu(n)$ is the number of summands in the most efficient way of writing n as a sum of numbers of the form 2^i-1; that is, $\mu(n)=\min\{m : n=\sum_{i=1}^{n}(2^{k_i}-1)$ for some integers $k_i\}$. The following generalization follows immediately from Theorem 3.1:

Corollary 4.1. $\mathrm{ex}(\chi T(i;\, 2^{j+1}-1)) = 2^{i+1}-1 \;\; (=(2^{i+1}-1)\mu(2^{j+1}-1))$.

In view of its consequences in the study of hit monomials, described below, Corollary 4.1 leads to

Conjecture 4.2. $\mathrm{ex}(\chi T(i;\, a)) = (2^{i+1}-1)\mu(a)$.

The case $a=2$, i arbitrary, would follow from Conjecture 1.2. Using a computer, Bruner has verified Conjecture 4.2 for all pairs (i, a) such that $a\le 31$ and $|T(i;\, a)|\le 255$.

The connection of Conjecture 4.2 with hit monomials is as follows: Recall that $\alpha(n)$ denotes the number of 1's in the binary expansion of the integer n. In [W] Wood extends and proves a conjecture due to Peterson.

Theorem 4.3 [W]. *Let M be a monomial of degree $|M| = d$, and suppose that e of its exponents are odd. If $\alpha(d+e) > e$, then M is hit.*

The proof involves writing M in the form $EF^2 = E \cdot \langle f \rangle (F)$, where E is squarefree of degree e and $f = |F|$, and showing that modulo hit elements, $E \cdot \theta(F) \equiv \chi\theta(E) \cdot F$ for any $\theta \in \mathscr{A}(2)$. The result then follows from the fact that $\mathrm{ex}(\chi\langle f \rangle) = \mu(f)$; one must check that the assumption $\alpha(d+e) > e$ implies that $\mu(f) > e$.

Singer has conjectured a generalization of Theorem 4.3, using a condition which involves not the degree of the squarefree part of M but rather the degree of the 2^{k+1}-powerfree part. That is, write $M = EF^{2^{k+1}}$ where E contains no 2^{k+1}-powers, and let $e = |E|$, $f = |F|$. The conjecture can be paraphrased in part as follows.

Conjecture 4.4 [Si]. *Suppose that M is a monomial with decomposition $M = EF^{2^{k+1}}$ as above. If $e < (2^{k+1} - 1)\mu(f)$, then M is hit.*

This time, we have that

$$EF^{2^{k+1}} = E \cdot [\langle 2^k f \rangle \cdots \langle 2f \rangle \langle f \rangle (F)] = E \cdot [T(k; f)(F)],$$

which modulo hit elements is congruent to $[\chi T(k; f)(E)] \cdot F$; but this element vanishes if $\mathrm{ex}(\chi T(k; f)) > |E|$, so we see that Conjecture 4.2 implies Conjecture 4.4. In the cases for which Conjecture 4.2 has been verified, we can state the following.

Theorem 4.5. *Suppose that M is a monomial with decomposition $EF^{2^{k+1}}$ as above and that $e < (2^{k+1} - 1)\mu(f)$. If $f = 2$ or $f = 2^i - 1$ for some i, or if (k, f) satisfies $f \leq 31$ and $(2^{k+1} - 1)f < 255$, then M is hit.*

In unpublished work, Singer has verified Conjecture 4.4 for $k = 1$ and all f using different techniques.

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Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109
E-mail address: Judith.Silverman@math.lsa.umich.edu