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THE SYLOW p-SUBGROUPS OF SEMICOMPLETE NILPOTENT GROUPS

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ABSTRACT. A nilpotent group whose group of outer automorphisms is trivial may contain elements of finite order. This paper is concerned with how large the Sylow p-subgroups of such a group can be. We show that in many cases the Sylow p-subgroups of such a semicomplete nilpotent group are always finite.

1. INTRODUCTION

The relation between the structure of a group G and that of its automorphism group $\mathrm{Aut}\, G$ has been the object of considerable research. Of particular interest in recent years are those groups which are *semicomplete.* A group G is said to be *complete* if it has trivial center and $\mathrm{Aut}\, G = \mathrm{Inn}\, G$. A group G is said to be semicomplete if $\mathrm{Aut}\, G = \mathrm{Inn}\, G$. Abelian groups of order greater than two are never semicomplete. Interest in automorphisms of nilpotent groups was spurred some years ago by a conjecture of Haimo and Schenkman that a nilpotent group could never be semicomplete. Results of Gaschütz [7, 8] and Zalesskiĭ [14] combined to verify the conjecture for periodic nilpotent groups. Shortly thereafter, however, Zalesskiĭ [15] constructed an example of a torsionfree nilpotent group that was semicomplete. Subsequent examples were provided by Heineken [9] and Robinson [10]. These groups were all torsionfree. Semicomplete nilpotent groups were further studied in [1–4]. In [3] semicomplete mixed nilpotent groups of class 2 were constructed.

Fournelle began the investigation of mixed semicomplete nilpotent groups in [2] by considering the special case in which the central quotient $G/Z(G)$ is torsionfree. He found in this case that either G is torsionfree or $G = H \times C_2$ where H is a torsionfree 2-radicable semicomplete nilpotent group and C_n is the cyclic group of order n. (Here we call a group G *p-radicable* if given $x \in G$ there exists $y \in G$ such that $y^p = x$. Then G is radicable if and only if G is p-radicable for all primes p. If G is also abelian we use the term *divisible* rather than radicable.) The aim of this paper is to continue this investigation. In

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particular we shall be concerned with the following problem, posed by Fournelle: If G is semicomplete, are its Sylow p-subgroups necessarily finite? We have been unable to solve this problem completely, there being one case which seems to be particularly difficult to handle. Let Z denote the center of G and let $Q = G/Z$. Our main theorem is the following:

Theorem A. *Let G be a semicomplete nilpotent group and p a prime. If one of the following conditions holds then the Sylow p-subgroup of G is finite.*

(i) $Q/Q'Q^p$ *is finite.*
(ii) Q_p, *the Sylow p-subgroup of Q, is countable.*
(iii) Q_p *has finite exponent.*
(iv) G *is nilpotent of class* 2.

This result extends certain results of [3] and the main result of [2]. The reader should also compare this result with [5, Theorem A] and note that Fournelle [3] has constructed uncountable semicomplete nilpotent groups. Furthermore, as far as we are aware, no examples of semicomplete nilpotent groups of class 3 or more have yet been constructed. Such examples would be of some interest. Our result cannot be strengthened further since Fournelle [3] gives examples where the torsion subgroup is infinite.

Part (i) of Theorem A takes on added significance when contrasted with the following result, which we shall also prove.

Theorem B. *Let G be a nilpotent group with central quotient* $Q = G/Z$. *If* $Q/Q'Q^p$ *is uncountable for some prime p then G is not semicomplete.*

Thus if G is semicomplete, $|Q/Q'Q^p| \leq \aleph_0$ for all primes p.

The layout of the paper is as follows. In §2 we prove the key result that the Sylow p-subgroup of the center of a semicomplete nilpotent group is reduced. In §3 we show that, if $\pi(G)$ denotes the set of primes p for which G possesses an element of order p, then the set $\pi(G) - \pi(G/Z(G))$ is somewhat restricted. Section 4 contains the proof of Theorem B. In §5 we prove Theorem A.

We make frequent use of a number of facts which are particularly useful when applied to nilpotent groups. For a group N we denote the terms of its upper central series by $Z_i(N)$ or simply by Z_i, if $i \geq 0$, and then there is a monomorphism

$$Z_{i+2}(N)/Z_{i+1}(N) \to \mathrm{Hom}(N_{\mathrm{ab}}, Z_{i+1}(N)/Z_i(N)).$$

Here N_{ab} denotes the abelianization of N. We denote the terms of the lower central series of N by $\gamma_i(N)$, so there is an epimorphism

$$N_{\mathrm{ab}} \otimes (\gamma_i(N)/\gamma_{i+1}(N)) \to \gamma_{i+1}(N)/\gamma_{i+2}(N) \quad \text{for all } i.$$

The reader is referred to [11] and [13] for further details of these maps. We shall also use the fact that if A is a central subgroup of the group N then we can identify $\mathrm{Hom}((N/A)_{\mathrm{ab}}, A)$ with the subgroup of Aut N which acts trivially on A and N/A. We also require a large number of facts concerning abelian groups, particularly properties of the functors Hom and $\otimes$, all of which can be found in Fuchs [6, Volume 1]. Our notation, when not explained, is standard. We shall denote the Sylow p-subgroup of a group G by G_p.

2. Z_p IS REDUCED

We begin with the following elementary result which is no doubt well known, so we omit its proof.

2.1. **Lemma.** *Let p be a prime and $\mathfrak{m}_1, \mathfrak{m}_2, \ldots$ be a sequence of cardinals such that $\mathfrak{m}_n \neq 0$ for infinitely many n. Then $\prod_{n=1}^{\infty} \prod_{\mathfrak{m}_n} C_{p^n}$ has infinite torsionfree rank.*

It is important to see how properties of the abelianization of a nilpotent group affect the whole group. Our next lemma shows how abelian subgroups are affected. We let $r_0(G)$ and $r_p(G)$ denote the torsionfree rank and p-rank respectively of the abelian group G.

2.2. **Lemma.** *Let N be a nilpotent group of class c. Let A be an abelian subgroup of N and let $r = r_0(N_{\mathrm{ab}})$. Then $r_0(A) \leq r + r^2 + \cdots + r^c$.*

Proof. Let N_i, for $i = 1, \ldots, c+1$, be the terms of the lower central series. It follows from our remarks in the introduction that N_i/N_{i+1} is of torsionfree rank at most r^i. If we set $A_i = A \cap N_i$ then A_i/A_{i+1} has torsionfree rank at most r^i and hence A has torsionfree rank as claimed in the lemma. □

We shall require a theorem of Fuchs concerning the structure of $\mathrm{Hom}(A, C_{p^\infty})$ for abelian groups A, which may be found in [6, Theorem 47.1]. For ease of reference we record this here.

2.3. **Theorem.** *Let p be a prime. Let A be an abelian group and let B be a p-basic subgroup of A where $B = \bigoplus_{n=0}^{\infty} B_n$ with $B_0 \cong \bigoplus_{\mathfrak{m}_0} \mathbb{Z}$ and $B_n \cong \bigoplus_{\mathfrak{m}_n} C_{p^n}$ for $n \geq 1$. Also let $\mathfrak{m} = r_p(A/B)$ and let $\mathfrak{t} = r_0(A/B)$. Then*

$$\mathrm{Hom}(A, C_{p^\infty}) \cong \prod_{\mathfrak{m}_0} C_{p^\infty} \oplus \prod_{n=1}^{\infty} \prod_{\mathfrak{m}_n} C_{p^n} \oplus \prod_{\mathfrak{m}} J_p \oplus \prod_{\mathfrak{t}\aleph_0} \mathbb{Q}$$

(*where J_p is the ring of p-adic integers and $\mathbb{Q}$ is the additive group of rational numbers*).

We use this result and its notation in the following lemma.

2.4. **Lemma.** *Let G be a semicomplete nilpotent group and suppose that p is a prime. Suppose $C_{p^\infty} \leq Z_p$, the p-component of $Z(G)$. Then*

(i) *Q_{ab} has finite torsionfree rank.*
(ii) *The p-component of Q_{ab} is bounded.*
(iii) *If B is a p-basic subgroup of Q_{ab} then Q_{ab}/B is a p'-group.*

Proof. The hypothesis $C_{p^\infty} \leq Z_p$ implies $H = \mathrm{Hom}(Q_{\mathrm{ab}}, C_{p^\infty})$ is a direct summand of $\mathrm{Hom}(Q_{\mathrm{ab}}, Z) \cong Z(Q)$. The structure of H is given by 2.3, with B p-basic in Q_{ab}.

Note that $r_0(Q_{\mathrm{ab}}) = \mathfrak{m}_0 + \mathfrak{t}$. If $\mathfrak{t}$ is infinite and $\mathfrak{t} \geq \mathfrak{m}_0$, then $r_0(H) \leq \mathfrak{t}$ by 2.2. On the other hand, $\prod_{\mathfrak{t}\aleph_0} \mathbb{Q} \leq H$, so H has torsionfree rank at least $2^{\mathfrak{t}}$, a contradiction. If $\mathfrak{m}_0$ is infinite and $\mathfrak{m}_0 \geq \mathfrak{t}$ then $r_0(H) \leq \mathfrak{m}_0$. Also,

$$r_0(H) \geq r_0\Big(\prod_{\mathfrak{m}_0} C_{p^\infty}\Big) = r_0\Big(\bigoplus_{2^{\mathfrak{m}_0}} \big(C_{p^\infty} \bigoplus \mathbb{Q}\big)\Big) \geq 2^{\mathfrak{m}_0},$$

by [6, p. 105], again a contradiction. Hence $\mathfrak{t}$ and $\mathfrak{m}_0$ are both finite. Thus $r_0(Q_{\mathrm{ab}})$ and $r_0(H)$ are finite. However, $\prod_{\mathfrak{t}\aleph_0} \mathbb{Q} \le H$, so $\mathfrak{t} = 0$. Hence Q_{ab}/B is periodic. Furthermore $\mathfrak{m} = 0$ since $\prod_{\mathfrak{m}} J_p \le H$ and J_p has infinite torsionfree rank. Hence Q_{ab}/B is a p'-group.

By 2.1, $r_0(H)$ finite also implies $\mathfrak{m}_n \ne 0$ for only finitely many n. Thus the torsion subgroup of B has exponent equal to p^k where k is the largest value of n for which $\mathfrak{m}_n \ne 0$. Since Q_{ab}/B is a p'-group it then follows that $(Q_{\mathrm{ab}})_p$ is the Sylow p-subgroup of B and thus $(Q_{\mathrm{ab}})_p$ is bounded. This completes the proof. □

At this stage we need the following result.

2.5. **Lemma.** *Suppose that A is an abelian group and p is a prime. Suppose*:

(i) *A has finite torsionfree rank.*
(ii) *A_p is bounded.*
(iii) *B is a p-basic subgroup and A/B is a p'-group.*

Then A has no images with unbounded p-component.

Proof. By (ii) and the purity of A_p, it follows that $A = A_p \oplus K$, for some subgroup K. We claim that K has no infinite p-groups as quotients from which the result follows immediately. Suppose that K/R is an infinite p-group for some R and let L be of maximal torsionfree rank in a p-basic subgroup of K. Then $(L+R)/R$ is a p-group so $L/L \cap R$ is finite and we may assume $L \le R$. Then K/R is a p'-group by (iii), whence $K = R$, a contradiction. □

For future reference we state the following fact, the proof of which is straightforward.

2.6. **Lemma.** *Suppose that A is an abelian group and $C \le A$. Suppose every image of C and every image of A/C has bounded p-component. Then every image of A has bounded p-component.*

We can now prove:

2.7. **Lemma.** *Suppose A and C are abelian groups and p is a prime. Suppose also that all images of A and C have bounded p-component and if L is a p-basic subgroup of C then C/L is a p'-group. Then all images of $A \otimes C$ have bounded p-component.*

Proof. By hypothesis, A and C must themselves have bounded p-component and also p-basic subgroups of A and C must have finite torsionfree rank. Since A_p and C_p are pure bounded subgroups of A and C respectively, we have $A = A_p \oplus M$ and $C = C_p \oplus N$, say, where M and N are p-torsionfree. Then

$$A \otimes C = (A_p \otimes C_p) \oplus (A_p \otimes N) \oplus (M \otimes C_p) \oplus (M \otimes N). \tag{1}$$

Since A_p and C_p are bounded p-groups, $A_p \otimes C_p$, $A_p \otimes N$, and $M \otimes C_p$ are also bounded p-groups and thus these three summands have no unbounded p-images.

Let B and D be p-basic subgroups of M and N respectively. By [6, Theorem 60.3], the exact sequences $0 \to B \to M \to M/B \to 0$ and $0 \to D \to N \to N/D \to 0$ yield an exact sequence

$$(B \otimes N) \oplus (D \otimes M) \to M \otimes N \to M/B \otimes N/D \to 0. \tag{2}$$

Now $N/D \cong C/L$ is a p'-group and hence $M/B \otimes N/D$ is a p'-group and consequently only has p' images. Since B and D have finite ranks, say k and l respectively, it follows that $(B \otimes N) \oplus (D \otimes M) = (\bigoplus_{i=1}^{k} N) \oplus (\bigoplus_{i=1}^{l} M)$. The hypotheses now imply that $(B \otimes N) \oplus (D \otimes M)$ has no quotients with unbounded p-component. The conclusion of the lemma now follows from the exact sequence (2), Lemma 2.6, and the direct sum decomposition (1). □

We use these technical results to establish the following lemma.

2.8. **Lemma.** *Let G be a semicomplete nilpotent group and suppose p is a prime. Suppose that $C_{p^\infty} \leq Z_p$. Then each lower central factor $\gamma_i Q/\gamma_{i+1} Q$ of Q has bounded p-component.*

Proof. The results 2.4 and 2.5 establish that Q_{ab} satisfies the hypotheses for both A and C in 2.7. Repeated application of 2.7 yields that $\bigotimes^i Q_{\mathrm{ab}}$ has no images with unbounded p-component. The result then follows using the epimorphisms mentioned in the introduction. □

We may now prove the main result of this section.

2.9. **Proposition.** *Suppose that G is a semicomplete nilpotent group and that p is a prime. Then the Sylow p-subgroup of the center of G is reduced.*

Proof. Suppose that Z_p contains a C_{p^∞}. It follows from 2.1, 2.3, and 2.4 that the subgroup $H = \mathrm{Hom}(Q_{\mathrm{ab}}, C_{p^\infty})$ is simply

$$H = \bigoplus_{\mathfrak{m}_0} C_{p^\infty} \oplus \left(\bigoplus_{n=1}^{k} \prod_{\mathfrak{m}_n} C_{p^n} \right),$$

with $\mathfrak{m}_0 = r_0(B) = r_0(Q_{\mathrm{ab}})$. Let D be the maximal divisible subgroup of H so $D \cong \bigoplus_{\mathfrak{m}_0} C_{p^\infty}$, and define $D_i = \gamma_i Q \cap D$. Then, by 2.8, the p-group D_i/D_{i+1} is of bounded exponent for all i. Hence $\mathfrak{m}_0 = 0$. Therefore Q_{ab}, and hence $Q \cong \mathrm{Aut}\, G$, is periodic. A theorem of Robinson [12, Theorem B] may now be applied to show that the torsion subgroup of G has finite exponent, which contradicts our assumption. Hence Z_p is reduced, as required. □

The following result is easily established using [11, Lemma 9.35].

2.10. **Corollary.** *If G is a semicomplete nilpotent group then $(Z_{i+1}/Z_i)_p$ is reduced for all i.*

It is now easy to establish the next result.

2.11. **Corollary.** *If G is a semicomplete nilpotent group then $(G/Z)_p$ contains no radicable subgroups.*

3. The case $p \in \pi(G) - \pi(G/Z)$

We begin this section with the following lemma, the proof of which we omit since it is essentially that of Lemma 3.1 in [2].

3.1. **Lemma.** *Suppose that G is a nilpotent group and $Z(G)$ contains an element of order p, for some prime p. Suppose that $p \notin \pi(\mathrm{Aut}\, G)$. Then either*

(i) *G is p-radicable, or*
(ii) *$G = C_p \times H$ for some p-radicable group H.*

This has the following corollary.

3.2. **Corollary.** *If G is a semicomplete nilpotent group and* $p \in \pi(G) - \pi(G/Z)$ *then either*

(i) *G is p-radicable, or*
(ii) $p = 2$ *and* $G = C_2 \times H$ *for some semicomplete 2-radicable group H.*

Note that in the second case $Z(G) = Z(H) \times C_2$ and thus $Q = G/Z(G) \cong H/Z(H)$, so $H/Z(H)$ is 2-torsionfree. We can now use the results of §2 to obtain the following:

3.3. **Proposition.** *Suppose that G is a semicomplete nilpotent group and that p is a prime in* $\pi(G) - \pi(G/Z)$. *Then either* G_p *is trivial or* $p = 2$ *and* G_p *is cyclic of order* 2.

Proof. It follows from 3.2 that either G is p-radicable or $p = 2$ and $G = C_2 \times H$, for some 2-radicable group H. In either case $Q = G/Z$ is p-radicable. In the first case, if $x \in Z_p(G)$ then there exists $y \in G$ such that $y^p = x$. Hence y is a p-element of Z since $p \notin \pi(G/Z)$. Therefore, $y \in Z_p(G)$. A similar argument holds in the second case also. Hence either $Z_p(G)$ or $Z_p(H)$ is divisible. The result now follows from 2.9. □

4. $Q/Q'Q^p$ UNCOUNTABLE

In the remainder of this paper we shall assume that $p \in \pi(G/Z)$. In [14] Zalesskiĭ defines a basic subgroup B of a nilpotent p-group N as the preimage of a basic subgroup of N_{ab}. Such basic subgroups have several properties which we shall exploit in this section. We shall show that for a semicomplete nilpotent group the size of $Q/Q'Q^p$ is somewhat restricted. First we prove

4.1. **Lemma.** *Suppose that N is a nilpotent group and p is a prime. If* $|N/N'N^p|$ *is infinite then* $|N/N^{p^n}| = |N/N'N^p|$ *for all natural numbers* n.

Proof. It follows from the epimorphisms mentioned in the introduction that $|N| = |N/N'|$, so it suffices to prove the result for abelian groups N. However, if B is a p-basic subgroup of N then $N/N^{p^n} \cong B/B^{p^n}$ for all n, by [6, p. 144(B)]. However, in this case, $|B| = |B/B^{p^n}|$ for all n. Hence the result. □

The key result in this section is the following. It is a generalization of [6, Corollary 34.4].

4.2. **Proposition.** *Suppose that N is a reduced nilpotent p-group for some prime p.*

(i) *If* $N/N'N^p$ *is finite then N is finite.*
(ii) *If* $|N/N'N^p| = \kappa$ *is infinite then* $\kappa \le |N| \le \kappa^{\aleph_0}$.

Proof. (i) follows from [13, Lemma 6.10].

(ii) Suppose that N has class c. We inductively construct subgroups R_i, D_i of N such that

(1) $D_{i-1}/D'_{i-1} = R_i/D'_{i-1} \oplus D_i/D'_{i-1}$.
(2) R_i/D'_{i-1} is reduced.
(3) D_i/D'_{i-1} is divisible.

(4) $\kappa \leq |R_i/D'_{i-1}| \leq \kappa^{\aleph_0}$.
(5) $D'_i \leq \gamma_{i+2}(N)$.

Let $D_0 = N$ and write N/N' as a direct sum of a divisible group D_1/D'_0 and a reduced group R_1/D'_0. Clearly $D'_0 \leq \gamma_2(N)$. If B_1/D'_0 is a basic subgroup of R_1/D'_0 then B_1 is a basic subgroup of D_0, so by [14, XIV]

$$(1) \qquad |B_1| = |N/N'N^p| = \kappa.$$

It follows from [6, Corollary 34.4] that $\kappa \leq |R_1/D'_0| \leq \kappa^{\aleph_0}$. Note in particular that (1) implies that $|N'| \leq \kappa$.

Suppose that we have constructed R_i and D_i, satisfying (1)–(5). Let $D_i/D'_i = R_{i+1}/D'_i \oplus D_{i+1}/D'_i$ with D_{i+1}/D'_i divisible and R_{i+1}/D'_i reduced. Let B_{i+1}/D'_i be a basic subgroup of R_{i+1}/D'_i. Then B_{i+1} is a basic subgroup of D_i, so

$$|B_{i+1}| = |D_i/D'_iD^p_i| = |D_i/D^p_i|.$$

However, $D_i = D^p_iD'_{i-1}$, so $D_i/D^p_i \cong D'_{i-1}/(D'_{i-1} \cap D^p_i)$. This has cardinality at most $|N'|$, which is at most κ. Hence by [6, Corollary 34.4], $\kappa \leq |R_{i+1}/D'_i| \leq \kappa^{\aleph_0}$. Also D_{i+1}/D'_i is divisible and $D_{i+1}/\gamma_3(D_i)$ is abelian-by-divisible, so [14, XI] implies $D_{i+1}/\gamma_3(D_i)$ is abelian. Hence,

$$D'_{i+1} \leq \gamma_3(D_i) \leq \gamma_3(\gamma_{i+2}N) \leq \gamma_{i+3}(N),$$

and the induction proceeds.

Since $D'_{c-1} \leq \gamma_{c+1}(N) = 1$, and since N is reduced, $D_c = 1$. Hence $D_{c-1} = R_c$. It then follows that $\kappa \leq |N| \leq \kappa^{\aleph_0}$. This completes the proof. □

The next lemma will prove to be repeatedly useful. For cardinal numbers $\mathfrak{m}, \mathfrak{n}$, let

$$d(\mathfrak{m}, \mathfrak{n}) = \begin{cases} \mathfrak{m}\mathfrak{n} & \text{if } \mathfrak{m} \text{ is finite,} \\ \mathfrak{n}^{\mathfrak{m}} & \text{if } \mathfrak{m} \text{ is infinite and } \mathfrak{n} \neq 1, \\ 2^{\mathfrak{m}} & \text{if } \mathfrak{m} \text{ is infinite and } \mathfrak{n} = 1. \end{cases}$$

4.3. **Lemma.** *Suppose G is a semicomplete nilpotent group. Let $r = r(Q/Q'Q^p)$ and let $t = r_p(Z)$. Then $(Z_2/Z_1)_p$ contains a subgroup of rank $d(r, t)$.*

Proof. Since $Q/Q'Q^p$ is an epimorphic image of Q_{ab}, there is a monomorphism

$$\mathrm{Hom}(Q/Q'Q^p, Z) \to \mathrm{Hom}(Q_{ab}, Z) \cong Z_2/Z_1.$$

Since $t = r(Z[p])$, we have

$$\mathrm{Hom}(Q/Q'Q^p, Z) \cong \mathrm{Hom}\left(\bigoplus_r C_p, Z\right) \cong \prod_r Z[p] \cong \prod_r \bigoplus_t C_p,$$

which has rank $d(r, t)$, as required. □

Blackburn has shown that if N is a nilpotent group of class c and p is a prime then there is a positive integer $f(p, c)$ such that, for every $n \geq f(p, c)$, every product of p^nth powers of elements of N is a $p^{n-f(p,c)}$th power (see [13, Theorem 6.4]). Hence if $x \in N_p$ and $x \notin {N_p}^p$, then $x \notin N^{p^{1+f(p,c)}}$.

Proof of Theorem B. Suppose $|Q/Q'Q^p| = \mathfrak{m} > \aleph_0$. By the remarks above and 4.1 we have

$$|Q_p/Q^p_p| \leq |Q/Q^{p^{1+f(p,c)}}| = |Q/Q'Q^p|.$$

It follows from 2.11 that Q_p is reduced and 4.3 shows Q_p is not finite. Hence $|Q_p/Q_p^p| = \kappa$ is infinite and 4.2 shows that $\kappa \leq |Q_p| \leq \kappa^{\aleph_0}$. However, by 4.3 again, $|Q_p| \geq 2^{\mathfrak{m}} > 2^{\aleph_0}$, so $\kappa \neq \aleph_0$. Thus $|Q_p| = \kappa = |Q_p/Q_p^p|$. This is a contradiction since $\kappa \leq \mathfrak{m}$. □

5. The proof of Theorem A

We note first that if Q_{ab} is p-divisible then $(Z_2/Z_1)[p] \cong \text{Hom}(Q_{\text{ab}}, Z)[p] = 0$ by [6, §43(E)], contradicting the fact that $p \in \pi(Q)$. Hence $Q/Q'Q^p \neq 0$. Our next result is almost immediate from what we have done already.

5.1. **Proposition.** *Suppose that G is a semicomplete nilpotent group. If Q_p is finite then G_p is finite.*

Proof. It follows from 4.3 and the fact that Q_p is finite that $r(Z_p)$ is finite. Hence Z_p is finite, since it is reduced. Thus G_p is finite. □

We now have

Proof of Theorem A. (a) Assume that $Q/Q'Q^p$ is finite. Then by 4.2(i) it follows that Q/Q^p is finite. As in the proof of Theorem B, Q_p/Q_p^p is also finite, so, by 4.2(i) again, Q_p is finite and the result now follows by 5.1.

(b) Suppose that Q_p is countable. By 5.1, we may assume that Q_p is infinite. Let $\mathfrak{m} = |Q/Q'Q^p|$. By 4.3, if $\mathfrak{m}$ is infinite then $|Q_p| \geq 2^{\mathfrak{m}}$, a contradiction. Hence $\mathfrak{m}$ is finite and the result follows from part (a).

(c) By part (a) and Theorem B we need only consider the case $|Q/Q'Q^p| = \aleph_0$. Let the exponent of Q_p be p^n. Then by 4.1 and the remarks preceding the proof of Theorem B, $|Q_p| = |Q_p/Q_p^{p^n}| \leq |Q/Q'Q^p| = \aleph_0$. The result now follows from part (b).

(d) Suppose that G is nilpotent of class 2. In this case $Q \cong \text{Aut}\, G \cong \text{Hom}(Q, Z)$, and by the preceding results we only need consider the situation when $|Q_p| > \aleph_0$ but $|Q/pQ| = \aleph_0$. (We revert here to additive notation for Q.) As before we have $|Q_p/pQ_p| \leq |Q/pQ| = \aleph_0$. Also $\aleph_0 < |Q_p| \leq 2^{\aleph_0}$. Since Q_p is reduced, it follows from [6, Corollary 27.3] that $Q = C_{p^n} \oplus X$, for some n and some X. Suppose that Z_p has infinite rank $\mathfrak{t}$. Then

$$\begin{aligned} Q &\cong \text{Hom}(Q, Z) = \text{Hom}(C_{p^n}, Z) \oplus \text{Hom}(X, Z) \\ &\cong Z[p^n] \oplus K \cong \bigoplus_{\mathfrak{t}} C_{p^{n(i)}} \oplus K \end{aligned}$$

where $K = \text{Hom}(X, Z)$ and $n(i) \leq n$ for all i. Hence

$$\begin{aligned} Q &\cong \text{Hom}(Q, Z) \cong \text{Hom}\left(\bigoplus_{\mathfrak{t}} C_p{}^{n(i)}, Z\right) \oplus \text{Hom}(K, Z) \\ &\cong \prod_{\mathfrak{t}} Z[p^{n(i)}] \oplus \text{Hom}(K, Z). \end{aligned}$$

Now $\prod_{\mathfrak{t}} Z[p^{n(i)}]$ has exponent bounded by p^n and if we set $L = \text{Hom}(K, Z)$ then we have

$$Q = \bigoplus_{2^{\mathfrak{t}}} C_p{}^{m(i)} \oplus L, \qquad m(i) \leq n \text{ for all } i.$$

Substituting again into $Q \cong \mathrm{Hom}(Q, Z)$ we obtain a p-subgroup of Q of cardinality $2^{2^{\mathfrak{t}}}$, contradicting the fact that $|Q_p| \leq 2^{\aleph_0}$. Thus $\mathfrak{t}$ is finite and so Z_p is finite since it is reduced. Hence Q_p is of bounded exponent and the desired result follows from part (c). This completes the proof of Theorem A. □

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