

ON THE EXISTENCE OF PERIODIC SOLUTIONS
FOR SCALAR SECOND ORDER DIFFERENTIAL EQUATIONS
WHEN ONLY THE ASYMPTOTIC BEHAVIOUR
OF THE POTENTIAL IS KNOWN

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ABSTRACT. Assuming only asymptotic conditions on the potential function, we prove the existence of periodic solutions for equations whose nonlinearity stays below the first curve of Fučík's spectrum.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

In this note we consider the periodic problem

$$(P) \quad \begin{cases} x'' + g(x) = e(t), \\ x(0) - x(T) = x'(0) - x'(T) = 0, \end{cases}$$

where $g : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and $e : [0, T] \rightarrow \mathbb{R}$ is measurable and bounded. We define $G(x) = \int_0^x g(s)ds$, a primitive of $g(x)$, and $\bar{e} = \frac{1}{T} \int_0^T e(t)dt$, the mean value of $e(t)$.

In our first result we assume the nonlinearity to lie, roughly speaking, between the first eigenvalue and the first curve of Fučík's spectrum.

Theorem 1. *Assume that*

$$\mu := \liminf_{x \rightarrow +\infty} \frac{2G(x)}{x^2} \quad \nu := \lim_{x \rightarrow -\infty} \frac{2G(x)}{x^2}$$

exist and are finite and positive. If

$$\frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}} > T,$$

then problem (P) has a solution.

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As a consequence of Theorem 1, we have that, if the limit

$$l := \lim_{|x| \rightarrow \infty} \frac{2G(x)}{x^2}$$

exists and $0 < l < (\frac{2\pi}{T})^2$, there exists a solution to (P). It is not known whether such a result still holds when l lies between any two eigenvalues of the associated linear operator.

Our second result deals with a one-sided assumption on the potential G . A further condition then has to be added in order to avoid resonance.

Theorem 2. *Assume that*

$$(i) \quad \liminf_{x \rightarrow +\infty} \frac{2G(x)}{x^2} < \left(\frac{\pi}{T}\right)^2;$$

$$(ii) \quad \lim_{|x| \rightarrow \infty} (G(x) - \bar{e}x) = +\infty.$$

Then problem (P) has a solution.

Notice that the above statements make use of assumptions relying only on the potential function G , and we do not require explicit conditions on the nonlinearity g , like monotonicity, sign conditions or growth restrictions (see [3–5, 8–10] and references therein).

The proof of Theorem 1 combines Leray-Schauder topological degree arguments with the upper and lower solutions method. To prove Theorem 2 we also need a variational setting. We will develop some ideas from Gossez and Omari [8, 9], Fernandes and Zanolin [5], and De Figueiredo and Ruf [2].

In §3, we will also state a necessary and sufficient nonresonance condition for problem (P), when assumption (i) of Theorem 2 holds. Needless to say, the symmetrical versions of our results hold as well, the assumptions at $\pm\infty$ being interchangeable.

Without loss of generality, we will consider from now on the case $\bar{e} = 0$.

2. PROOF OF THEOREM 1

We first remark that, since μ and ν are positive, the function g is unbounded from below and from above on \mathbb{R} . Since e is bounded, there exist A and B in \mathbb{R} such that, for a.e. $t \in [0, T]$,

$$(1) \quad g(A) < e(t) < g(B).$$

We will use some arguments from [8]. If $A > B$, the result is a consequence of the lower and upper solutions method (cf. [1, 8]). Assume then $A < B$. Replacing x by $x - \frac{A+B}{2}$, it is of no loss of generality to suppose

$$(2) \quad A < 0 < B.$$

Moreover, we can assume the existence of a constant $c_1 > 0$ such that, for every $s \in \mathbb{R}$,

$$(3) \quad \operatorname{sgn}(s)g(s) > -c_1.$$

In fact, otherwise one could find $A' > B$ with the same property as A , and the lower and upper solutions method would apply again.

Now we fix a $\theta \in]0, (\frac{\pi}{T})^2[$ and we consider, for $\lambda \in]0, 1[$, the problem

$$(P_\lambda) \quad \begin{cases} x'' + \lambda\theta x + (1 - \lambda)g(x) = (1 - \lambda)e(t), \\ x(0) - x(T) = x'(0) - x'(T) = 0. \end{cases}$$

The existence of a solution to (P) will be assured by the Leray-Schauder theory if we are able to find an open bounded subset \mathcal{A} of the space $C(0, T)$, containing 0, whose boundary does not contain any of the solutions of (P_λ) , $\lambda \in]0, 1[$ (cf. [10]). We claim that there exist two constants R and S such that $R < A < B < S$, and the set

$$\mathcal{A} = \{x \in C(0, T) \mid \min(x) \in]R, B[, \max(x) \in]A, S[\}$$

is the one we are looking for. The boundary of set \mathcal{A} is

$$\partial\mathcal{A} = \{x \in C(0, T) \mid \min(x) \in [R, B], \max(x) \in [A, S] \text{ and either } \min(x) \in \{R, B\} \text{ or } \max(x) \in \{A, S\}\}.$$

If x is a solution of (P_λ) , it cannot be that $\min(x) = B$ or $\max(x) = A$. This can easily be seen by writing the equation in (P_λ) at the points where x attains its minimum and maximum and taking into account (1) and (2).

Notice that, if $x \in \partial\mathcal{A}$,

$$(4) \quad \exists t_x \in [0, T] : x(t_x) \in [A, B].$$

We will find $S > B$ such that $\max(x) \neq S$ for every solution x of (P_λ) having the property (4). Take $\mu' > \mu$ in such a way that $(\pi/\sqrt{\mu'}) + (\pi/\sqrt{\nu}) > T$, and define $\hat{e}(t) = e(t) + c_1$, $\hat{g}(x) = g(x) + c_1$, and correspondingly $\hat{G}(x) = G(x) + c_1x$. Then it is easy to see that

$$\limsup_{x \rightarrow +\infty} \left[\mu' \frac{x^2}{2} - \hat{G}(x) \right] = +\infty,$$

and we can find a sequence (S_n) such that $S_n \rightarrow +\infty$ as $n \rightarrow \infty$ and

$$(5) \quad \forall s \in [0, S_n[, \quad \mu' \frac{s^2}{2} - \hat{G}(s) < \mu' \frac{S_n^2}{2} - \hat{G}(S_n).$$

We will show that we can take $S = S_{\bar{n}}$ for \bar{n} large enough. In fact, suppose by contradiction that there exist two sequences (x_n) , (λ_n) such that $\lambda_n \in]0, 1[$, x_n is a solution of (P_{λ_n}) , $\max(x_n) = S_n$, and $\exists t_n \in [0, T] : x_n(t_n) \in [A, B]$.

First we prove that $m_n := \min(x_n) \rightarrow -\infty$ as $n \rightarrow \infty$. Suppose by contradiction that, for a subsequence, (m_n) is bounded from below. Then, by (3), there is a constant $c_2 > 0$ such that

$$\lambda_n \theta x_n(t) + (1 - \lambda_n)g(x_n(t)) > -c_2$$

for every $t \in [0, T]$. Since

$$\int_0^T [\lambda_n \theta x_n + (1 - \lambda_n)g(x_n)] = 0,$$

multiplying the equation in (P_{λ_n}) by $\tilde{x}_n = x_n - \frac{1}{T} \int_0^T x_n(t) dt$ and integrating, we get

$$\begin{aligned}
 \int_0^T (\tilde{x}'_n)^2 &= \int_0^T [\lambda_n \theta x_n + (1 - \lambda_n)g(x_n) + c_2] \tilde{x}_n - \int_0^T [c_2 + (1 - \lambda_n)e] \tilde{x}_n \\
 (6) \quad &\leq \|\tilde{x}_n\|_{L^\infty} \left\{ \int_0^T [\lambda_n \theta x_n + (1 - \lambda_n)g(x_n) + c_2] + c_2 T + \|e\|_{L^1} \right\} \\
 &\leq \|\tilde{x}_n\|_{L^\infty} [2c_2 T + \|e\|_{L^1}].
 \end{aligned}$$

Then, (\tilde{x}_n) is bounded, and this contradicts the fact that $\max(x_n) \rightarrow +\infty$ and $\exists t_n \in [0, T] : x_n(t_n) \in [A, B]$. This proves that $m_n \rightarrow -\infty$ as $n \rightarrow \infty$.

Extending our functions by T-periodicity, we can then find two intervals $[\alpha_n, \beta_n]$ and $[\gamma_n, \delta_n]$ containing, respectively, a point of minimum and a point of maximum of x_n , such that

$$\begin{aligned}
 (7) \quad &x_n(\alpha_n) = x_n(\beta_n) = x_n(\gamma_n) = x_n(\delta_n) = 0, \\
 &x_n(t) < 0 \quad \text{for } t \in]\alpha_n, \beta_n[, \\
 &x_n(t) > 0 \quad \text{for } t \in]\gamma_n, \delta_n[, \\
 &(\beta_n - \alpha_n) + (\delta_n - \gamma_n) \leq T.
 \end{aligned}$$

We will now use some ideas from [5, 6, 11]. We first restrict our analysis to the interval $[\gamma_n, \delta_n]$. Because of (3), we have that $\hat{g}(s) > 0$ for all $s > 0$. We can define, for $S > 0$, the time-map

$$\hat{\tau}(S) = \sqrt{2} \int_0^S \frac{d\xi}{\sqrt{\hat{G}(S) - \hat{G}(\xi)}}.$$

It has been shown in [6] that the time-map $\hat{\tau}(S_n)$ is a good estimate, as $n \rightarrow \infty$, for the length of $[\delta_n, \gamma_n]$. Since, by (5), $\hat{\tau}(S_n) \geq (\pi/\sqrt{\mu'})$ for every n , we obtain

$$(8) \quad \liminf_{n \rightarrow \infty} (\delta_n - \gamma_n) \geq \frac{\pi}{\sqrt{\mu'}}.$$

On the other hand, one can estimate the length of $[\alpha_n, \beta_n]$ in the following way. One defines $\check{e}(t) = e(t) - c_1$, $\check{g}(x) = g(x) - c_1$, and correspondingly $\check{G}(x) = G(x) - c_1 x$. Then, for $m < 0$, the time-map is defined by

$$\check{\tau}(m) = \sqrt{2} \int_m^0 \frac{d\xi}{\sqrt{\check{G}(m) - \check{G}(\xi)}}.$$

Using the fact that ν is the limit of the quotient $2G(x)/x^2$, one can prove that

$$\liminf_{m \rightarrow -\infty} \check{\tau}(m) \geq \frac{\pi}{\sqrt{\nu}}$$

(cf. [11, Corollary 8]). Since $\check{\tau}(m_n)$ is a good estimate, as $n \rightarrow \infty$, for the length of $[\alpha_n, \beta_n]$, we have that

$$\liminf_{n \rightarrow \infty} (\beta_n - \alpha_n) \geq \frac{\pi}{\sqrt{\nu}}.$$

We get a contradiction with (7) and the fact that $(\pi/\sqrt{\mu'}) + (\pi/\sqrt{\nu}) > T$. So we can choose $S = S_{\bar{n}}$, with \bar{n} large enough.

Finally, we prove that there is $R < A$ such that, if x is a solution of (P_λ) , $\lambda \in]0, 1[$, satisfying $\max(x) \leq S$ and (4), then $\min(x) > R$. In fact, if $\max(x) \leq S$, there exists a constant $c_3 > 0$ such that

$$\lambda \theta x(t) + (1 - \lambda)g(x(t)) < c_3.$$

Multiplying the equation in (P_λ) by \tilde{x} and integrating, we obtain an estimate analogous to (6). Hence $\|\tilde{x}\|_{L^\infty}$ is bounded by a constant depending only on S , and since x satisfies (4), there is $R < A$ such that $\min(x) > R$.

Thus we showed that no solution of (P_λ) , $\lambda \in]0, 1[$, can lie on $\partial\mathcal{A}$, and the proof is complete.

3. PROOF OF THEOREM 2

If g is unbounded from above and below on \mathbb{R} , one can proceed as in the proof of Theorem 1. The only difference lies in the contradiction with (7), which is already reached in (8), since $\mu' < (\frac{\pi}{T})^2$. In the spirit of [8], we then have the following.

Corollary 1. *Assuming that*

$$\liminf_{x \rightarrow +\infty} \frac{2G(x)}{x^2} < \left(\frac{\pi}{T}\right)^2,$$

problem (P) has a solution for every $e \in L^\infty(0, T)$ if and only if g is unbounded from above and below.

Now assume g to be bounded either from below or above; let us treat the first case, the other being similar. So, assume there exists $c_4 > 0$ such that, for all $s \in \mathbb{R}$,

$$(9) \quad g(s) \geq -c_4.$$

We will prove the existence of a solution for (P) by finding a critical point of the associated functional $\phi : H_T^1 \rightarrow \mathbb{R}$ defined as

$$\phi(x) = \int_0^T \left[\frac{1}{2}(x'(t))^2 - G(x(t)) + e(t)x(t) \right] dt.$$

We will show that the functional ϕ has a mountain-pass geometry and satisfies the Palais-Smale condition.

Define the set $\mathcal{S} = \{x \in H_T^1 : x \leq 0 \text{ and } \exists t_x \in [0, T] : x(t_x) = 0\}$. Extending our functions by periodicity, for every $x \in \mathcal{S}$ we can write $\phi(x)$ as an integral over the interval $[t_x, t_x + T]$. Because of (9), we have that $\lim_{x \rightarrow -\infty} \frac{2G(x)}{x^2} = 0$, and it is easy to see, by Poincaré's inequality, that ϕ is bounded below on \mathcal{S} . On the other hand, (ii) implies that $(-\phi)$ is coercive on the space of constant functions. We can then find $r > 0$ sufficiently large

for which

$$\max\{\phi(-r), \phi(r)\} < \inf_{\mathcal{S}} \phi.$$

Since, by continuity, every path in $\Gamma = \{\gamma \in C([-r, r], H_T^1) : \gamma(\pm r) = \pm r\}$ has to cross \mathcal{S} , we have

$$\inf_{\mathcal{S}} \phi \leq \inf \sup_{\Gamma} \phi.$$

We are then in a mountain-pass geometry.

In order to conclude, we now prove the Palais-Smale condition. Let (x_n) be a sequence in H_T^1 such that $(\phi(x_n))$ is bounded and $\phi'(x_n) \rightarrow 0$, as $n \rightarrow \infty$. Then there is a constant $c_5 > 0$ such that, for every $u \in H_T^1$,

$$(10) \quad \left| \int_0^T [x_n' u' - g(x_n)u + eu] dt \right| < c_5 \|u\|_{H^1}.$$

Taking $u \equiv 1$, we get

$$\left| \int_0^T g(x_n(t)) dt \right| \leq c_5 \sqrt{T},$$

hence, by (9),

$$\int_{[g \geq 0]} g(x_n(t)) dt \leq c_5 \sqrt{T} - \int_{[g < 0]} g(x_n(t)) dt \leq c_5 \sqrt{T} + c_4 T.$$

Consequently, there is a constant c_6 such that

$$\int_0^T |g(x_n(t))| dt \leq c_6.$$

Taking $u = \tilde{x}_n$ in (10) permits us to conclude that (\tilde{x}_n) has to be bounded. Assume by contradiction that, for a subsequence, $\|x_n\|_{H^1} \rightarrow \infty$. Then $|x_n(t)| \rightarrow \infty$ uniformly in t . But this is in contradiction with the fact that (\tilde{x}_n) and $(\phi(x_n))$ are bounded, since G is coercive. So (x_n) is bounded, and the Palais-Smale condition holds (see [12]).

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