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DECOMPOSITION OF PEANO DERIVATIVES

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ABSTRACT. Let Δ' be the class of all derivatives, and let $[\Delta']$ be the vector space generated by Δ' and O'Malley's class B_1^*. In [1] it is shown that every function in $[\Delta']$ is of the form $g' + hk'$, where g, h, and k are differentiable, and that $f \in [\Delta']$ if and only if there is a sequence of derivatives v_n and closed sets A_n such that $\bigcup_{n=1}^{\infty} A_n = \mathbf{R}$ and $f = v_n$ on A_n. The sequence of sets A_n together with the corresponding functions v_n is called a decomposition of f. In this paper we show that every Peano derivative belongs to $[\Delta']$. Also we show that for Peano derivatives the sets A_n can be chosen to be perfect.

1. INTRODUCTION

Let C be the family of all continuous functions on $\mathbf{R}$, Δ the family of all differentiable functions on $\mathbf{R}$ and Δ' the family of all derivatives on $\mathbf{R}$. If Γ is a family of functions defined on $\mathbf{R}$, then by $[\Gamma]$ we denote the family of all functions f on $\mathbf{R}$ with the following property: for every $n = 1, 2, \ldots$ there exist $v_n \in \Gamma$ and closed sets A_n such that $\bigcup_{n=1}^{\infty} A_n = \mathbf{R}$ and $f = v_n$ on A_n. In [1, Theorem 2] it is shown that the following four conditions are equivalent:

(i) There are g, h, and k in Δ such that $h', k' \in [C]$ and $f = g' + hk'$.
(ii) There is a $\varphi \in \Delta'$ and $\psi \in [C]$ such that $f = \varphi + \psi$.
(iii) $f \in [\Delta']$.
(iv) There is a dense open set T and a function $k \in \Delta$ such that f is a derivative on T and $f = k'$ on $\mathbf{R} \setminus T$.

Statement (ii) implies that $[\Delta']$ is a vector space generated by Δ' and $[C]$. In [1, Theorem 3] it is shown that each approximate derivative, each approximately continuous function, and each function in $B_1^* = [C]$ belongs to the class $[\Delta']$. In [5] O'Malley showed that for approximate derivatives sets A_n from the definition of $[\Delta']$ can be chosen to be perfect. A question raised in [1] is: "Does every Peano derivative belong to $[\Delta']$?"

The main goal of this paper is to show that a kth Peano derivative is in $[\Delta']$ and that the sets A_n from the definition of $[\Delta']$ can be chosen to be perfect. We will prove even more, namely, that a kth Peano derivative is a composite derivative of the $(k-1)$th Peano derivative. An immediate consequence of this result is that a kth Peano derivative is an approximate derivative of the

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$(k-1)$th Peano derivative almost everywhere. This result was first proved by Zygmund and Marcinkiewicz (see [12, p. 77]).

2. Preliminaries

In this section we will recall the definition of Peano derivatives and some known properties of Peano derivatives.

Definition 1. Let f be a continuous function defined on $\mathbf{R}$. We say that f has kth Peano derivative at some point x, if there are real numbers $f_1, f_2, \ldots, f_k$ such that

$$(1)\quad f(x+h) = f(x) + hf_1 + \cdots + h^k \frac{f_k}{k!} + h^k \varepsilon_k(h) \quad \text{where } \lim_{h\to 0} \varepsilon_k(h) = 0.$$

The number f_k is called the kth Peano derivative of f at x, and since it depends only on a function f and a point x, it will be convenient to denote it by $f_k(x)$. Similarly the continuous function $\varepsilon_k(h)$ depends on x, so we may denote it by $\varepsilon_k(x, h)$. Also, it will be convenient to denote $f(x)$ by $f_0(x)$. With this notation (1) becomes $f(x+h) = \sum_{j=0}^{k} h^j \frac{f_j(x)}{j!} + h^k \varepsilon_k(x, h)$. From Definition 1 it is easy to see that if the kth Peano derivative exists, so does the nth, for $1 \le n < k$.

It is known that the kth Peano derivative is Baire 1, Darboux, and has Denjoy property. For these, and some other properties of Peano derivatives, see [3, 4, 6, 8–12].

3. Formula

In this section we will derive a formula that is the crux of the proof of Theorem 2.

Theorem 1. *Let f be a continuous function on $\mathbf{R}$, and let $x \neq x_1$ and $t \neq 0$ be points such that $f_k(x)$ and $f_k(x_1)$ exist. Then the following formula holds:*

$$\begin{aligned}
&\frac{f_{k-1}(x_1) - f_{k-1}(x)}{x_1 - x} - f_k(x) \\
&= \frac{t}{x_1 - x}\frac{k-1}{2} f_k(x) \\
&\quad + \sum_{j=0}^{k-1} (-1)^{k-1-j} \binom{k-1}{j} \frac{(x_1 - x + jt)^k}{t^{k-1}(x_1 - x)} \varepsilon_k(x, x_1 - x + jt) \\
&\quad - \frac{t}{x_1 - x} \left\{ \frac{k-1}{2} f_k(x_1) + \sum_{j=0}^{k-1} (-1)^{k-1-j} \binom{k-1}{j} j^k \varepsilon_k(x_1, jt) \right\}.
\end{aligned}$$

To prove this formula we need some lemmas.

Lemma 1. *For $m \in \mathbf{N}$ we have*

$$\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} j^i = \begin{cases} 0 & \text{if } i = 0, 1, \ldots, m-1, \\ m! & \text{if } i = m, \\ \frac{m}{2}(m+1)! & \text{if } i = m+1. \end{cases}$$

Proof. The case $0 \leq i \leq m$ is a well-known exercise in mathematical induction. So let us consider only the case $i = m + 1$. Let

$$a(m) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} j^{m+1}.$$

Then we have the following recursive formula: $a(m) = ma(m-1) + mm!$, and since $a(1) = 1$, we have $a(m) = \frac{m}{2}(m+1)!$ $\square$

Definition 2. Let f be a function defined on $\mathbf{R}$. Then for any two points x_1 and t let

$$\Delta_{k-1} = \sum_{j=0}^{k-1} (-1)^{k-1-j} \binom{k-1}{j} f(x_1 + jt).$$

Lemma 2. *Let f be a function defined on $\mathbf{R}$ having a kth Peano derivative at some point x_1. Then for any t the following holds:*

$$\Delta_{k-1} = t^{k-1} f_{k-1}(x_1) + t^k \frac{k-1}{2} f_k(x_1) + t^k \sum_{j=0}^{k-1} (-1)^{k-1-j} \binom{k-1}{j} j^k \varepsilon_k(x_1, jt).$$

Proof.

$$\begin{aligned}
\Delta_{k-1} &= \sum_{j=0}^{k-1} (-1)^{k-1-j} \binom{k-1}{j} \left(\sum_{l=0}^{k} (jt)^l \frac{f_l(x_1)}{l!} + (jt)^k \varepsilon_k(x_1, jt) \right) \\
&= \sum_{l=0}^{k} \left(\sum_{j=0}^{k-1} (-1)^{k-1-j} \binom{k-1}{j} j^l \right) t^l \frac{f_l(x_1)}{l!} \\
&\quad + t^k \sum_{j=0}^{k-1} (-1)^{k-1-j} \binom{k-1}{j} j^k \varepsilon_k(x_1, jt).
\end{aligned}$$

The rest follows from Lemma 1. $\square$

Lemma 3. *Let f be a function on $\mathbf{R}$, and let $x \neq x_1$ be a point such that $f_k(x)$ exists. Then*

$$\begin{aligned}
\Delta_{k-1} &= t^{k-1} f_{k-1}(x) + t^{k-1}(x_1 - x) f_k(x) + t^k \frac{k-1}{2} f_k(x) \\
&\quad + \sum_{j=0}^{k-1} (-1)^{k-1-j} \binom{k-1}{j} (x_1 - x + jt)^k \varepsilon_k(x, x_1 - x + jt).
\end{aligned}$$

Proof.

$$\Delta_{k-1} = \sum_{j=0}^{k-1}(-1)^{k-1-j}\binom{k-1}{j}$$
$$\cdot\left(\sum_{l=0}^{k}(x_1-x+jt)^l\frac{f_l(x)}{l!}+(x_1-x+jt)^k\varepsilon_k(x, x_1-x+jt)\right)$$
$$=\sum_{j=0}^{k-1}(-1)^{k-1-j}\binom{k-1}{j}\sum_{l=0}^{k}(x_1-x+jt)^l\frac{f_l(x)}{l!}$$
$$+\sum_{j=0}^{k-1}(-1)^{k-1-j}\binom{k-1}{j}(x_1-x+jt)^k\varepsilon_k(x, x_1-x+jt).$$

Since

$$\sum_{j=0}^{k-1}(-1)^{k-1-j}\binom{k-1}{j}\sum_{l=0}^{k}(x_1-x+jt)^l\frac{f_l(x)}{l!}$$
$$=\sum_{j=0}^{k-1}(-1)^{k-1-j}\binom{k-1}{j}\sum_{l=0}^{k}\sum_{i=0}^{l}\binom{l}{i}(x_1-x)^{l-i}(jt)^i\frac{f_l(x)}{l!}$$
$$=\sum_{l=0}^{k}\sum_{i=0}^{l}\binom{l}{i}(x_1-x)^{l-i}t^i\left(\sum_{j=0}^{k-1}(-1)^{k-1-j}\binom{k-1}{j}j^i\right)\frac{f_l(x)}{l!},$$

by Lemma 1 it is equal to

$$\sum_{l=k-1}^{k}\sum_{i=k-1}^{l}\binom{l}{i}(x_1-x)^{l-i}t^i\left(\sum_{j=0}^{k-1}(-1)^{k-1-j}\binom{k-1}{j}j^i\right)\frac{f_l(x)}{l!}.$$

Applying Lemma 1 once more it is equal to

$$t^{k-1}f_{k-1}(x)+t^{k-1}(x_1-x)f_k(x)+t^k\frac{k-1}{2}f_k(x). \quad \square$$

Proof of Theorem 1. The assertion follows directly from Lemmas 2 and 3. □

4. Decomposition

In this section we will prove the main theorem, namely, that a kth Peano derivative belongs to $[\Delta']$.

Definition 3. Suppose that a function f has a kth Peano derivative at every point of $\mathbf{R}$. Let

$$H(f, M, \delta)$$
$$=\left\{x:\left|\frac{k-1}{2}f_k(x)+\sum_{j=0}^{k-1}(-1)^{k-1-j}\binom{k-1}{j}j^k\varepsilon_k(x, jt)\right|\le M \text{ for } |t|<\delta\right\}$$

where M and δ are some positive constants.

Theorem 2. *$H = H(f, M, \delta)$ is closed and f_{k-1} is differentiable on H relative to H with $(f_{k-1}|_H)'(x) = f_k(x)$, also $|f_k(x)| \le 2M$ for every $x \in H$.*

Proof. Let $x \in \overline{H}$. Let $1 > \varepsilon > 0$ be given. There is $0 < \eta < \delta$ such that $|\varepsilon_k(x, h)| < \varepsilon$ whenever $|h| < \eta$. Let $x_n \in H$ so that $|x_n - x| < \eta/k$. Then for $t = (x_n - x)\varepsilon^{1/k}$ we have $|t| < \delta$ and $|x_n - x + jt| < \eta$, for $j = 0, 1, \dots, k-1$. Then the formula from Theorem 1 gives

$$\begin{aligned}
&\left|\frac{f_{k-1}(x_n) - f_{k-1}(x)}{x_n - x} - f_k(x)\right| \\
&\quad \le \varepsilon^{1/k}\frac{k-1}{2}|f_k(x)| + \sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(1+j\varepsilon^{1/k})^k}{\varepsilon^{(k-1)/k}}|\varepsilon_k(x, x_n - x + jt)| \\
&\qquad + \varepsilon^{1/k}\left|\frac{k-1}{2}f_k(x_n) + \sum_{j=0}^{k-1}(-1)^{k-1-j}\binom{k-1}{j} j^k \varepsilon_k(x_n, jt)\right| \\
&\quad \le \varepsilon^{1/k}\frac{k-1}{2}|f_k(x)| + \sum_{j=0}^{k-1}\binom{k-1}{j}(1+j\varepsilon^{1/k})^k\varepsilon^{1/k} + \varepsilon^{1/k}M.
\end{aligned}$$

Therefore as $x_n \to x$ with $x_n \in H$ we get

$$\frac{f_{k-1}(x_n) - f_{k-1}(x)}{x_n - x} - f_k(x) \to 0.$$

Now let $x \in \overline{H}$, $\{x_n\}$ a sequence in H with $x_n \to x$, and $0 \ne |t| < \delta$. Then the formula from Theorem 1 yields

$$\begin{aligned}
&\left|t\left(\frac{k-1}{2}f_k(x) + \sum_{j=0}^{k-1}(-1)^{k-1-j}\binom{k-1}{j}\frac{(x_n - x + jt)^k}{t^k}\varepsilon_k(x, x_n - x + jt)\right)\right| \\
&\quad \le \left|t\left(\frac{k-1}{2}f_k(x_n) + \sum_{j=0}^{k-1}(-1)^{k-1-j}\binom{k-1}{j} j^k\varepsilon_k(x_n, jt)\right)\right| \\
&\qquad + |f_{k-1}(x_n) - f_{k-1}(x) - (x_n - x)f_k(x)| \\
&\quad \le |t|M + |f_{k-1}(x_n) - f_{k-1}(x) - (x_n - x)f_k(x)|.
\end{aligned}$$

Letting $n \to \infty$ the left-hand side becomes

$$\left|t\left(\frac{k-1}{2}f_k(x) + \sum_{j=0}^{k-1}(-1)^{k-1-j}\binom{k-1}{j} j^k\varepsilon_k(x, jt)\right)\right|,$$

while the right-hand side is $|t|M$. Hence $x \in H$.

That $|f_k(x)| \le 2M$ on H follows from the definition of H taking $t = 0$. □

Lemma 4. $\bigcup_{M=1}^{\infty} H(f, M, 1) = \mathbf{R}$.

Proof. The assertion follows from the fact that $\varepsilon_k(x, jt)$ is a continuous function of t for $j = 0, 1, \dots, k-1$. □

The following corollary is an immediate consequence of Lemma 4 and Theorem 2.

Corollary 1. *Let* f *be a continuous function on* $\mathbf{R}$ *such that* $f_k(x)$ *exists at every point of* $\mathbf{R}$. *Then* $f_k \in [\Delta']$.

Proof. The corollary follows directly from Theorem 2, Lemma 4, and the fact that for any function g defined on a closed set P, that is differentiable with respect to P, there is a function G differentiable on $\mathbf{R}$ so that $G|_P = g$ and $G'|_P = g'$. (See Mařik [7].) □

Definition 4. Let f be a function defined on $\mathbf{R}$. If there exist a function g and closed sets A_n, $n = 1, 2, \ldots$, such that $\bigcup_{n=1}^{\infty} A_n = \mathbf{R}$ and $g|'_{A_n}(x) = f(x)$ for $x \in A_n$, then we say that f is a composite derivative of g.

Corollary 2. f_k *is a composite derivative of* f_{k-1}.

An immediate consequence of this result is the following corollary, first proved by Zygmund and Marcinkiewicz. (See Zygmund [12, p. 77].)

Corollary 3. f_k *is the approximate derivative of* f_{k-1} *almost everywhere.*

5. On $(k-1)$th Peano derivatives

It was known that for any point x there is a sequence $x_n \to x$ so that

$$\lim_{n\to\infty} (f_{k-1}(x_n) - f_{k-1}(x))/(x_n - x) = f_k(x).$$

(See Weil [11] or Mařik [6].) In this section we will prove that if f_k exists at some point x and f_{k-1} exists at some neighborhood of the point x, then there is a perfect set P of positive measure such that x is a bilateral point of accumulation of P and f_{k-1} differentiates at x along P with $f_{k-1}|'_P(x) = f_k(x)$. In order to prove the above we need a few lemmas, two of which (Lemma 5 and Lemma 7) are known. (See Corominas [3].)

Lemma 5. *Let* f *and* g *be functions on* $\mathbf{R}$ *such that the* n*th Peano derivatives* $f_n(x)$ *and* $g_n(x)$ *exist at some point* x. *Then the function* fg *has an* n*th Peano derivative at* x *and*

$$(fg)_n(x) = \sum_{j=0}^{n} \binom{n}{j} f_j(x) g_{n-j}(x).$$

Lemma 6. *Let* f *and* g *be functions on* $\mathbf{R}$ *such that the* n*th Peano derivative,* $f_n(x)$, *and the* n*th ordinary derivative,* $g^{(n)}(x)$, *exist at some point* x. *Then*

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} (fg^{(j)})_{n-j}(x) = f_n(x) g(x).$$

Proof. By Lemma 5

$$\begin{aligned}
\sum_{j=0}^{n}(-1)^j \binom{n}{j} (fg^{(j)})_{n-j}(x) & \\
&= \sum_{j=0}^{n}(-1)^j \binom{n}{j} \sum_{i=0}^{n-j} \binom{n-j}{i} f_i(x)(g^{(j)})_{(n-j-i)}(x) \\
&= \sum_{j=0}^{n}(-1)^j \binom{n}{j} \sum_{i=0}^{n-j} \binom{n-j}{i} f_i(x)g^{(n-i)}(x) \\
&= \sum_{i=0}^{n}\sum_{j=0}^{n-i}(-1)^j \binom{n}{j} \binom{n-j}{i} f_i(x)g^{(n-i)}(x) \\
&= \sum_{i=0}^{n}\sum_{j=0}^{n-i}(-1)^j \binom{n-i}{j} \binom{n}{i} f_i(x)g^{(n-i)}(x) \\
&= \sum_{i=0}^{n}\binom{n}{i}\sum_{j=0}^{n-i}(-1)^j \binom{n-i}{j} f_i(x)g^{(n-i)}(x) \\
&= f_n(x)g(x) + \sum_{i=0}^{n-1}\binom{n}{i}(1-1)^{n-i}f_i(x)g^{(n-i)}(x) = f_n(x)g(x). \quad \square
\end{aligned}$$

Lemma 7. *Let H be a function defined in a neighborhood $\mathscr{O}$ of a point y. Suppose that H is n times Peano differentiable in $\mathscr{O}$ and that H_n is m times Peano differentiable in $\mathscr{O}$. Then H is $(n+m)$ times Peano differentiable at y, and $H_{(n+m)}(y) = (H_n)_m(y)$.*

Lemma 8. *Let f be defined in some neighborhood $\mathscr{O}$ of 0. Suppose that the kth Peano derivative of f at 0 exists and that the lth Peano derivative of f exists in $\mathscr{O}$, where k and l are positive integers with $l \le k-1$. Also suppose that $f(0) = f_1(0) = \cdots = f_k(0) = 0$. Let $g(y) = y^{-(k-l)}$. Then the function h defined by*

$$\begin{aligned}
h(y) = \binom{l}{0} f(y)g(y) - \binom{l}{1}\int_0^y f(t)g'(t)\,dt & \\
+\cdots+(-1)^l\binom{l}{l}\int_0^y\int_0^{x_2}\cdots\int_0^{x_{l-1}} f(t)g^{(l)}(t)\,dt\cdots dx_2 & \quad \textit{for } y \neq 0,
\end{aligned}$$

and $h(0) = 0$ has an lth Peano derivative on $\mathscr{O}$.

Moreover,

$$h_l(y) = \begin{cases} f_l(y)/y^{k-l} & \textit{if } y \neq 0, \\ 0 & \textit{if } y = 0. \end{cases}$$

Proof. By assumption, $f(y) = y^k \varepsilon_k(0, y)$. Consequently all of the above integrals are integrals of continuous functions. Hence h is well defined. Moreover, for $y \neq 0$, $y \in \mathscr{O}$

$$H(y) = \int_0^y \int_0^{x_2} \cdots \int_0^{x_{i-1}} f(t) g^{(i)}(t)\, dt \cdots dx_2, \qquad i = 1, \ldots, l,$$

is i times ordinarily differentiable and $H^{(i)}(y) = f(y) g^{(i)}(y)$ for $i = 1, \ldots, l$. By Lemma 5, $fg^{(i)}$ is l times Peano differentiable at y. Therefore by Lemma 7, H is l times Peano differentiable at y and

$$H_l(y) = (H^{(i)})_{l-i}(y) = (f(y) g^{(i)}(y))_{(l-i)}.$$

Hence h is l times Peano differentiable at y and

$$h_l(y) = \sum_{j=0}^{l} (-1)^j \binom{l}{j} (fg^{(j)})_{(l-j)}(y),$$

and, by Lemma 6, $h_l(y) = f_l(y) g(y)$.

It remains to prove that $h_l(0)$ exists and that $h_l(0) = 0$. For $y \neq 0$

$$\begin{aligned} \frac{h(y)}{y^l} = \frac{1}{y^l} \Bigg\{ \binom{l}{0} y^l \varepsilon_k(0, y) &+ (k-l) \binom{l}{1} \int_0^y t^{l-1} \varepsilon_k(0, t)\, dt \\ &+ \cdots + (k-l)(k-l+1) \cdots (k-1) \\ &\cdot \binom{l}{l} \int_0^y \int_0^{x_2} \cdots \int_0^{x_{l-1}} \varepsilon_k(0, t) dt \cdots dx_2 \Bigg\}. \end{aligned}$$

Hence $\lim_{y \to 0}(h(y)/y^l) = 0$. Therefore $h(0) = h_1(0) = \cdots = h_l(0) = 0$. □

Now suppose that f has an lth Peano derivative in some neighborhood $\mathscr{O}$ of a point x and that $f_k(x)$ exists. Consider a function $T(y) = f(y) - f(x) - (y-x) f_1(x) - \cdots - (y-x)^k f_k(x)/k!$ and its translate $G(t) = T(x+t)$.

Then G satisfies the hypothesis of Lemma 8 and by that lemma the function H defined by

$$\begin{aligned} H(y) = \binom{l}{0} G(y) g(y) &- \binom{l}{1} \int_0^y G(t) g'(t)\, dt \\ &+ \cdots + (-1)^l \binom{l}{l} \int_0^y \int_0^{x_2} \cdots \int_0^{x_{l-1}} G(t) g^{(l)}(t)\, dt \cdots dx_2 \quad \text{for } y \neq 0 \end{aligned}$$

and $H(0) = 0$ has an lth Peano derivative on $x - \mathscr{O}$.

Moreover, by the same lemma,

$$H_l(y) = \begin{cases} G_l(y)/y^{k-l} & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$$

But

$$G_l(t) = T_l(t+x) = f_l(t+x) - f_l(x) - t f_{l+1}(x) - \cdots - t^{k-l} \frac{f_k(x)}{(k-l)!}.$$

Therefore we have proved the following theorem.

Theorem 3. *Suppose that a function f in some neighborhood $\mathscr{O}$ of a point x has an lth Peano derivative in $\mathscr{O}$ and a kth Peano derivative at x, where*

$0 \leq l \leq k$. *Then the function* F *defined on* $\mathscr{O}$ *by*

$$F(y) = \begin{cases} (f_l(y) - \sum_{j=0}^{k-l}(y-x)^j(f_{l+j}(x))/j!)/(y-x)^{k-l} & \text{if } y \neq x, \\ 0 & \text{if } y = x \end{cases}$$

is an l*th Peano derivative.*

Corollary 4. *Suppose that a function* f *defined in some neighborhood* $\mathscr{O}$ *of a point* x *has a* $(k-1)$*th Peano derivative in* $\mathscr{O}$ *and* k*th Peano derivative at* x. *Then there exists a perfect set* $P \subset \mathscr{O}$ *of positive measure such that* x *is a bilateral point of accumulation of* P *and*

$$\lim_{y \in P, y \to x} \frac{f_{k-1}(y) - f_{k-1}(x)}{y - x} = f_k(x).$$

Proof. The function F from Theorem 3, applied with $l = k-1$ is a $(k-1)$th Peano derivative and hence Baire 1, Darboux, and has Denjoy property. Therefore, there is a perfect set P of positive measure such that x is a bilateral point of accumulation of P and such that F is continuous at x with respect to P. □

6. A_n CAN BE CHOSEN TO BE PERFECT

In this section we will prove that the sets A_n from the definition of $[\Delta']$ for Peano derivatives can be chosen to be perfect.

Let $y \in H(f, M, 1)$ be an isolated point of $H(f, M, 1)$. Then there is a $1 > \delta(y) > 0$ so that $(y - 2\delta(y), y + 2\delta(y)) \cap H(f, M, 1) = \{y\}$. Let P_y be a perfect set containing y so that y is a bilateral point of accumulation of P_y satisfying

$$\lim_{z \in P_y, z \to y} \frac{f_{k-1}(z) - f_{k-1}(y)}{z - y} = f_k(y)$$

and

$$\left| \frac{f_{k-1}(z) - f_{k-1}(y)}{z - y} - f_k(y) \right| \leq 1 \quad \text{for every } z \in P_y.$$

Corollary 4 assures the existence of P_y. If $P_y \cap (y + \frac{1}{n+1}, y + \frac{1}{n}) \neq \varnothing$, for $n \in \mathbf{Z} \setminus \{-1, 0\}$, then by the Baire category theorem there is $Q_n(y) \subset P_y \cap (y + \frac{1}{n+1}, y + \frac{1}{n})$, such that Q_n is perfect and that there is $M_n \in \mathbf{N}$ with $Q_n(y) \subset H(f, M_n, 1)$. Let

$$Q_y = \bigcup_{n \in \mathbf{Z} \setminus \{-1, 0\}} Q_n(y) \cap (y - \delta^2(y), y + \delta^2(y)) \cup \{y\},$$

and let

$$H_M = H(f, M, 1) \cup \{Q_y : y \in H(f, M, 1),\ y \text{ is isolated in } H(f, M, 1)\}.$$

Theorem 4. H_M *is a perfect set, and* f_{k-1} *is differentiable on* H_M *relative to* H_M *with* $(f_{k-1}|_{H_M})'(x) = f_k(x)$.

Proof. By the construction of H_M we see that no point is an isolated point. Note that each of Q_y is perfect and that $Q_y \cap Q_z = \varnothing$ if $y, z \in H(f, M, 1)$ are two different isolated points of $H(f, M, 1)$. Suppose that H_M is not closed. Then there is a sequence $\{z_n\}$ and a point z such that $\lim_{n\to\infty} z_n = z$

and $\{z_n\} \cap H(f, M, 1) = \varnothing$, but then either there is a subsequence $\{z_{n_k}\} \subset \{z_n\}$ and $y \in H(f, M, 1)$ with y an isolated point of $H(f, M, 1)$ so that $\{z_{n_k}\} \subset Q_y$, or there is a sequence $\{y_{n_k}\} \subset H(f, M, 1)$ so that y_{n_k} is an isolated point of $H(f, M, 1)$ and $z_{n_k} \in Q_{y_{n_k}}$ for $k = 1, 2, \ldots$. In the first case $z \in Q_y \subset H_M$, and in the second $y_{n_k} \to z$ and hence $z \in H(f, M, 1)$.

Now if $x \in H_M$ is an isolated point of $H(f, M, 1)$, then clearly f'_{k-1} at x relative to H_M exists and is equal to $f_k(x)$. If $x \in Q_y$ for some $y \in H(f, M, 1)$ where y is an isolated point of $H(f, M, 1)$, then there is $n \in \mathbf{Z}$ so that $x \in Q_n(y) \subset H(f, M_n(y), 1)$, and by the fact that there are two numbers $a < b$ so that $(a, b) \cap H_M = Q_n(y)$, we see that f'_{k-1} at x relative to H_M exists and is equal to $f_k(x)$.

Finally let $x \in H(f, M, 1)$, and x not an isolated point of $H(f, M, 1)$. Let $\varepsilon > 0$ be given. Then there is $\varepsilon > \eta > 0$ so that

$$\left|\frac{f_{k-1}(y) - f_{k-1}(x)}{y - x} - f_k(x)\right| < \varepsilon$$

whenever $y \in H(f, M, 1)$ and $|y - x| < \eta$.

Let y be an isolated point of $H(f, M, 1)$, and let $z \in Q_y$ with $|z - x| < \eta/2$. Since $|y - z| < \delta^2(y) < \delta(y)$ and $|y - x| > 2\delta(y)$, we have $\eta/2 > |x - z| \geq |x - y| - |y - z| > 2\delta(y) - \delta(y) = \delta(y)$. Hence $|y - x| \leq |y - z| + |z - x| < \delta(y) + \eta/2 < \eta$.

Now

$$\begin{aligned}
&\left|\frac{f_{k-1}(z) - f_{k-1}(x)}{z - x} - f_k(x)\right| \\
&\quad = \left|\left(\frac{f_{k-1}(y) - f_{k-1}(x)}{y - x} - f_k(x)\right)\frac{y - x}{z - x}\right. \\
&\qquad \left. + \left(\frac{f_{k-1}(z) - f_{k-1}(y)}{z - y} - f_k(y)\right)\frac{z - y}{z - x} + \frac{z - y}{z - x}(f_k(y) - f_k(x))\right| \\
&\quad \leq \left|\frac{f_{k-1}(y) - f_{k-1}(x)}{y - x} - f_k(x)\right|\left|1 - \frac{z - y}{z - x}\right| \\
&\qquad + \left|\frac{f_{k-1}(z) - f_{k-1}(y)}{z - y} - f_k(y)\right|\left|\frac{z - y}{z - x}\right| + \left|\frac{z - y}{z - x}\right|(|f_k(x)| + |f_k(y)|) \\
&\quad \leq \varepsilon\left(1 + \frac{\delta^2(y)}{\delta(y)}\right) + 1 \cdot \frac{\delta^2(y)}{\delta(y)} + \frac{\delta^2(y)}{\delta(y)}4M \\
&\quad \leq 2\varepsilon + \delta(y)(1 + 4M) \leq 2\varepsilon + \frac{\varepsilon}{2}(1 + 4M),
\end{aligned}$$

and since ε was arbitrary, we have that f'_{k-1} at x relative to H_M exists and equals $f_k(x)$. $\square$

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