PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 119, Number 2, October 1993

DECOMPOSITION OF PEANO DERIVATIVES

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(Communicated by Andrew Bruckner)

ABSTRACT. Let Δ' be the class of all derivatives, and let $[\Delta']$ be the vector space generated by Δ' and O'Malley's class B_1^*. In [1] it is shown that every function in $[\Delta']$ is of the form $g' + hk'$, where g, h, and k are differentiable, and that $f \in [\Delta']$ if and only if there is a sequence of derivatives v_n and closed sets A_n such that $\bigcup_{n=1}^{\infty} A_n = \mathbf{R}$ and $f = v_n$ on A_n. The sequence of sets A_n together with the corresponding functions v_n is called a decomposition of f. In this paper we show that every Peano derivative belongs to $[\Delta']$. Also we show that for Peano derivatives the sets A_n can be chosen to be perfect.

1. INTRODUCTION

Let C be the family of all continuous functions on $\mathbf{R}$, Δ the family of all differentiable functions on $\mathbf{R}$ and Δ' the family of all derivatives on $\mathbf{R}$. If Γ is a family of functions defined on $\mathbf{R}$, then by $[\Gamma]$ we denote the family of all functions f on $\mathbf{R}$ with the following property: for every $n = 1, 2, \ldots$ there exist $v_n \in \Gamma$ and closed sets A_n such that $\bigcup_{n=1}^{\infty} A_n = \mathbf{R}$ and $f = v_n$ on A_n. In [1, Theorem 2] it is shown that the following four conditions are equivalent:

(i) There are g, h, and k in Δ such that $h', k' \in [C]$ and $f = g' + hk'$.
(ii) There is a $\varphi \in \Delta'$ and $\psi \in [C]$ such that $f = \varphi + \psi$.
(iii) $f \in [\Delta']$.
(iv) There is a dense open set T and a function $k \in \Delta$ such that f is a derivative on T and $f = k'$ on $\mathbf{R} \setminus T$.

Statement (ii) implies that $[\Delta']$ is a vector space generated by Δ' and $[C]$. In [1, Theorem 3] it is shown that each approximate derivative, each approximately continuous function, and each function in $B_1^* = [C]$ belongs to the class $[\Delta']$. In [5] O'Malley showed that for approximate derivatives sets A_n from the definition of $[\Delta']$ can be chosen to be perfect. A question raised in [1] is: "Does every Peano derivative belong to $[\Delta']$?"

The main goal of this paper is to show that a kth Peano derivative is in $[\Delta']$ and that the sets A_n from the definition of $[\Delta']$ can be chosen to be perfect. We will prove even more, namely, that a kth Peano derivative is a composite derivative of the $(k-1)$th Peano derivative. An immediate consequence of this result is that a kth Peano derivative is an approximate derivative of the

Received by the editors March 4, 1992.
1991 *Mathematics Subject Classification*. Primary 26A24.

$(k-1)$th Peano derivative almost everywhere. This result was first proved by Zygmund and Marcinkiewicz (see [12, p. 77]).

2. Preliminaries

In this section we will recall the definition of Peano derivatives and some known properties of Peano derivatives.

Definition 1. Let f be a continuous function defined on $\mathbf{R}$. We say that f has kth Peano derivative at some point x, if there are real numbers $f_1, f_2, \dots, f_k$ such that

$$(1) \quad f(x+h) = f(x) + hf_1 + \cdots + h^k \frac{f_k}{k!} + h^k \varepsilon_k(h) \quad \text{where } \lim_{h\to 0} \varepsilon_k(h) = 0.$$

The number f_k is called the kth Peano derivative of f at x, and since it depends only on a function f and a point x, it will be convenient to denote it by $f_k(x)$. Similarly the continuous function $\varepsilon_k(h)$ depends on x, so we may denote it by $\varepsilon_k(x, h)$. Also, it will be convenient to denote $f(x)$ by $f_0(x)$. With this notation (1) becomes $f(x+h) = \sum_{j=0}^{k} h^j \frac{f_j(x)}{j!} + h^k \varepsilon_k(x, h)$. From Definition 1 it is easy to see that if the kth Peano derivative exists, so does the nth, for $1 \le n < k$.

It is known that the kth Peano derivative is Baire 1, Darboux, and has Denjoy property. For these, and some other properties of Peano derivatives, see [3, 4, 6, 8–12].

3. Formula

In this section we will derive a formula that is the crux of the proof of Theorem 2.

Theorem 1. *Let f be a continuous function on $\mathbf{R}$, and let $x \neq x_1$ and $t \neq 0$ be points such that $f_k(x)$ and $f_k(x_1)$ exist. Then the following formula holds*:

$$\begin{aligned}
&\frac{f_{k-1}(x_1) - f_{k-1}(x)}{x_1 - x} - f_k(x) \\
&= \frac{t}{x_1 - x}\frac{k-1}{2} f_k(x) \\
&\quad + \sum_{j=0}^{k-1} (-1)^{k-1-j} \binom{k-1}{j} \frac{(x_1 - x + jt)^k}{t^{k-1}(x_1 - x)} \varepsilon_k(x, x_1 - x + jt) \\
&\quad - \frac{t}{x_1 - x}\left\{ \frac{k-1}{2} f_k(x_1) + \sum_{j=0}^{k-1} (-1)^{k-1-j} \binom{k-1}{j} j^k \varepsilon_k(x_1, jt) \right\}.
\end{aligned}$$

To prove this formula we need some lemmas.

Lemma 1. *For $m \in \mathbf{N}$ we have*

$$\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} j^i = \begin{cases} 0 & \text{if } i = 0, 1, \dots, m-1, \\ m! & \text{if } i = m, \\ \frac{m}{2}(m+1)! & \text{if } i = m+1. \end{cases}$$

Proof. The case $0 \le i \le m$ is a well-known exercise in mathematical induction. So let us consider only the case $i = m+1$. Let

$$a(m) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} j^{m+1}.$$

Then we have the following recursive formula: $a(m) = ma(m-1) + mm!$, and since $a(1) = 1$, we have $a(m) = \frac{m}{2}(m+1)!$ $\square$

Definition 2. Let f be a function defined on $\mathbf{R}$. Then for any two points x_1 and t let

$$\Delta_{k-1} = \sum_{j=0}^{k-1} (-1)^{k-1-j} \binom{k-1}{j} f(x_1 + jt).$$

Lemma 2. *Let f be a function defined on $\mathbf{R}$ having a kth Peano derivative at some point x_1. Then for any t the following holds:*

$$\Delta_{k-1} = t^{k-1} f_{k-1}(x_1) + t^k \frac{k-1}{2} f_k(x_1) + t^k \sum_{j=0}^{k-1} (-1)^{k-1-j} \binom{k-1}{j} j^k \varepsilon_k(x_1, jt).$$

Proof.

$$\begin{aligned} \Delta_{k-1} &= \sum_{j=0}^{k-1} (-1)^{k-1-j} \binom{k-1}{j} \left(\sum_{l=0}^{k} (jt)^l \frac{f_l(x_1)}{l!} + (jt)^k \varepsilon_k(x_1, jt) \right) \\ &= \sum_{l=0}^{k} \left(\sum_{j=0}^{k-1} (-1)^{k-1-j} \binom{k-1}{j} j^l \right) t^l \frac{f_l(x_1)}{l!} \\ &\quad + t^k \sum_{j=0}^{k-1} (-1)^{k-1-j} \binom{k-1}{j} j^k \varepsilon_k(x_1, jt). \end{aligned}$$

The rest follows from Lemma 1. $\square$

Lemma 3. *Let f be a function on $\mathbf{R}$, and let $x \ne x_1$ be a point such that $f_k(x)$ exists. Then*

$$\begin{aligned} \Delta_{k-1} &= t^{k-1} f_{k-1}(x) + t^{k-1}(x_1 - x) f_k(x) + t^k \frac{k-1}{2} f_k(x) \\ &\quad + \sum_{j=0}^{k-1} (-1)^{k-1-j} \binom{k-1}{j} (x_1 - x + jt)^k \varepsilon_k(x, x_1 - x + jt). \end{aligned}$$

Proof.

$$\begin{aligned}\Delta_{k-1} &= \sum_{j=0}^{k-1}(-1)^{k-1-j}\binom{k-1}{j} \\ &\quad\cdot\left(\sum_{l=0}^{k}(x_1-x+jt)^l\frac{f_l(x)}{l!}+(x_1-x+jt)^k\varepsilon_k(x,x_1-x+jt)\right) \\ &= \sum_{j=0}^{k-1}(-1)^{k-1-j}\binom{k-1}{j}\sum_{l=0}^{k}(x_1-x+jt)^l\frac{f_l(x)}{l!} \\ &\quad+\sum_{j=0}^{k-1}(-1)^{k-1-j}\binom{k-1}{j}(x_1-x+jt)^k\varepsilon_k(x,x_1-x+jt).\end{aligned}$$

Since

$$\begin{aligned}&\sum_{j=0}^{k-1}(-1)^{k-1-j}\binom{k-1}{j}\sum_{l=0}^{k}(x_1-x+jt)^l\frac{f_l(x)}{l!} \\ &\quad=\sum_{j=0}^{k-1}(-1)^{k-1-j}\binom{k-1}{j}\sum_{l=0}^{k}\sum_{i=0}^{l}\binom{l}{i}(x_1-x)^{l-i}(jt)^i\frac{f_l(x)}{l!} \\ &\quad=\sum_{l=0}^{k}\sum_{i=0}^{l}\binom{l}{i}(x_1-x)^{l-i}t^i\left(\sum_{j=0}^{k-1}(-1)^{k-1-j}\binom{k-1}{j}j^i\right)\frac{f_l(x)}{l!},\end{aligned}$$

by Lemma 1 it is equal to

$$\sum_{l=k-1}^{k}\sum_{i=k-1}^{l}\binom{l}{i}(x_1-x)^{l-i}t^i\left(\sum_{j=0}^{k-1}(-1)^{k-1-j}\binom{k-1}{j}j^i\right)\frac{f_l(x)}{l!}.$$

Applying Lemma 1 once more it is equal to

$$t^{k-1}f_{k-1}(x)+t^{k-1}(x_1-x)f_k(x)+t^k\frac{k-1}{2}f_k(x). \quad \square$$

Proof of Theorem 1. The assertion follows directly from Lemmas 2 and 3. □

4. Decomposition

In this section we will prove the main theorem, namely, that a kth Peano derivative belongs to $[\Delta']$.

Definition 3. Suppose that a function f has a kth Peano derivative at every point of **R**. Let

$$\begin{aligned}&H(f,M,\delta) \\ &\quad=\left\{x:\left|\frac{k-1}{2}f_k(x)+\sum_{j=0}^{k-1}(-1)^{k-1-j}\binom{k-1}{j}j^k\varepsilon_k(x,jt)\right|\le M\ \text{for } |t|<\delta\right\}\end{aligned}$$

where M and δ are some positive constants.

Theorem 2. *$H = H(f, M, \delta)$ is closed and f_{k-1} is differentiable on H relative to H with $(f_{k-1}|_H)'(x) = f_k(x)$, also $|f_k(x)| \le 2M$ for every $x \in H$.*

Proof. Let $x \in \overline{H}$. Let $1 > \varepsilon > 0$ be given. There is $0 < \eta < \delta$ such that $|\varepsilon_k(x, h)| < \varepsilon$ whenever $|h| < \eta$. Let $x_n \in H$ so that $|x_n - x| < \eta/k$. Then for $t = (x_n - x)\varepsilon^{1/k}$ we have $|t| < \delta$ and $|x_n - x + jt| < \eta$, for $j = 0, 1, \dots, k-1$. Then the formula from Theorem 1 gives

$$\begin{aligned}
&\left|\frac{f_{k-1}(x_n) - f_{k-1}(x)}{x_n - x} - f_k(x)\right| \\
&\quad \le \varepsilon^{1/k}\frac{k-1}{2}|f_k(x)| + \sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(1+j\varepsilon^{1/k})^k}{\varepsilon^{(k-1)/k}}|\varepsilon_k(x, x_n - x + jt)| \\
&\qquad + \varepsilon^{1/k}\left|\frac{k-1}{2}f_k(x_n) + \sum_{j=0}^{k-1}(-1)^{k-1-j}\binom{k-1}{j}j^k\varepsilon_k(x_n, jt)\right| \\
&\quad \le \varepsilon^{1/k}\frac{k-1}{2}|f_k(x)| + \sum_{j=0}^{k-1}\binom{k-1}{j}(1+j\varepsilon^{1/k})^k\varepsilon^{1/k} + \varepsilon^{1/k}M.
\end{aligned}$$

Therefore as $x_n \to x$ with $x_n \in H$ we get

$$\frac{f_{k-1}(x_n) - f_{k-1}(x)}{x_n - x} - f_k(x) \to 0.$$

Now let $x \in \overline{H}$, $\{x_n\}$ a sequence in H with $x_n \to x$, and $0 \ne |t| < \delta$. Then the formula from Theorem 1 yields

$$\begin{aligned}
&\left|t\left(\frac{k-1}{2}f_k(x) + \sum_{j=0}^{k-1}(-1)^{k-1-j}\binom{k-1}{j}\frac{(x_n - x + jt)^k}{t^k}\varepsilon_k(x, x_n - x + jt)\right)\right| \\
&\quad \le \left|t\left(\frac{k-1}{2}f_k(x_n) + \sum_{j=0}^{k-1}(-1)^{k-1-j}\binom{k-1}{j}j^k\varepsilon_k(x_n, jt)\right)\right| \\
&\qquad + |f_{k-1}(x_n) - f_{k-1}(x) - (x_n - x)f_k(x)| \\
&\quad \le |t|M + |f_{k-1}(x_n) - f_{k-1}(x) - (x_n - x)f_k(x)|.
\end{aligned}$$

Letting $n \to \infty$ the left-hand side becomes

$$\left|t\left(\frac{k-1}{2}f_k(x) + \sum_{j=0}^{k-1}(-1)^{k-1-j}\binom{k-1}{j}j^k\varepsilon_k(x, jt)\right)\right|,$$

while the right-hand side is $|t|M$. Hence $x \in H$.

That $|f_k(x)| \le 2M$ on H follows from the definition of H taking $t = 0$. □

Lemma 4. $\bigcup_{M=1}^{\infty} H(f, M, 1) = \mathbf{R}$.

Proof. The assertion follows from the fact that $\varepsilon_k(x, jt)$ is a continuous function of t for $j = 0, 1, \dots, k-1$. □

The following corollary is an immediate consequence of Lemma 4 and Theorem 2.

Corollary 1. *Let f be a continuous function on $\mathbf{R}$ such that $f_k(x)$ exists at every point of $\mathbf{R}$. Then $f_k \in [\Delta']$.*

Proof. The corollary follows directly from Theorem 2, Lemma 4, and the fact that for any function g defined on a closed set P, that is differentiable with respect to P, there is a function G differentiable on $\mathbf{R}$ so that $G|_P = g$ and $G'|_P = g'$. (See Mařik [7].) □

Definition 4. Let f be a function defined on **R**. If there exist a function g and closed sets A_n, $n = 1, 2, \dots$, such that $\bigcup_{n=1}^{\infty} A_n = \mathbf{R}$ and $g|'_{A_n}(x) = f(x)$ for $x \in A_n$, then we say that f is a composite derivative of g.

Corollary 2. *f_k is a composite derivative of f_{k-1}.*

An immediate consequence of this result is the following corollary, first proved by Zygmund and Marcinkiewicz. (See Zygmund [12, p. 77].)

Corollary 3. *f_k is the approximate derivative of f_{k-1} almost everywhere.*

5. On $(k-1)$th Peano derivatives

It was known that for any point x there is a sequence $x_n \to x$ so that

$$\lim_{n\to\infty} (f_{k-1}(x_n) - f_{k-1}(x))/(x_n - x) = f_k(x).$$

(See Weil [11] or Mařik [6].) In this section we will prove that if f_k exists at some point x and f_{k-1} exists at some neighborhood of the point x, then there is a perfect set P of positive measure such that x is a bilateral point of accumulation of P and f_{k-1} differentiates at x along P with $f_{k-1}|'_P(x) = f_k(x)$. In order to prove the above we need a few lemmas, two of which (Lemma 5 and Lemma 7) are known. (See Corominas [3].)

Lemma 5. *Let f and g be functions on $\mathbf{R}$ such that the nth Peano derivatives $f_n(x)$ and $g_n(x)$ exist at some point x. Then the function fg has an nth Peano derivative at x and*

$$(fg)_n(x) = \sum_{j=0}^{n} \binom{n}{j} f_j(x) g_{n-j}(x).$$

Lemma 6. *Let f and g be functions on $\mathbf{R}$ such that the nth Peano derivative, $f_n(x)$, and the nth ordinary derivative, $g^{(n)}(x)$, exist at some point x. Then*

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} (fg^{(j)})_{n-j}(x) = f_n(x) g(x).$$

Proof. By Lemma 5

$$\begin{aligned}
\sum_{j=0}^{n}&(-1)^j \binom{n}{j} (fg^{(j)})_{n-j}(x) \\
&= \sum_{j=0}^{n}(-1)^j \binom{n}{j} \sum_{i=0}^{n-j} \binom{n-j}{i} f_i(x)(g^{(j)})_{(n-j-i)}(x) \\
&= \sum_{j=0}^{n}(-1)^j \binom{n}{j} \sum_{i=0}^{n-j} \binom{n-j}{i} f_i(x)g^{(n-i)}(x) \\
&= \sum_{i=0}^{n}\sum_{j=0}^{n-i}(-1)^j \binom{n}{j} \binom{n-j}{i} f_i(x)g^{(n-i)}(x) \\
&= \sum_{i=0}^{n}\sum_{j=0}^{n-i}(-1)^j \binom{n-i}{j} \binom{n}{i} f_i(x)g^{(n-i)}(x) \\
&= \sum_{i=0}^{n}\binom{n}{i}\sum_{j=0}^{n-i}(-1)^j \binom{n-i}{j} f_i(x)g^{(n-i)}(x) \\
&= f_n(x)g(x) + \sum_{i=0}^{n-1}\binom{n}{i}(1-1)^{n-i} f_i(x)g^{(n-i)}(x) = f_n(x)g(x). \quad \square
\end{aligned}$$

Lemma 7. *Let H be a function defined in a neighborhood $\mathscr{O}$ of a point y. Suppose that H is n times Peano differentiable in $\mathscr{O}$ and that H_n is m times Peano differentiable in $\mathscr{O}$. Then H is $(n+m)$ times Peano differentiable at y, and $H_{(n+m)}(y) = (H_n)_m(y)$.*

Lemma 8. *Let f be defined in some neighborhood $\mathscr{O}$ of 0. Suppose that the kth Peano derivative of f at 0 exists and that the lth Peano derivative of f exists in $\mathscr{O}$, where k and l are positive integers with $l \le k-1$. Also suppose that $f(0) = f_1(0) = \cdots = f_k(0) = 0$. Let $g(y) = y^{-(k-l)}$. Then the function h defined by*

$$\begin{aligned}
h(y) = \binom{l}{0} f(y)g(y) - \binom{l}{1}\int_0^y f(t)g'(t)\,dt \\
+ \cdots + (-1)^l \binom{l}{l} \int_0^y \int_0^{x_2} \cdots \int_0^{x_{l-1}} f(t)g^{(l)}(t)\,dt\cdots dx_2 \quad \text{for } y \neq 0,
\end{aligned}$$

and $h(0) = 0$ has an lth Peano derivative on $\mathscr{O}$.

Moreover,

$$h_l(y) = \begin{cases} f_l(y)/y^{k-l} & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$$

Proof. By assumption, $f(y) = y^k \varepsilon_k(0, y)$. Consequently all of the above integrals are integrals of continuous functions. Hence h is well defined. Moreover, for $y \neq 0$, $y \in \mathscr{O}$

$$H(y) = \int_0^y \int_0^{x_2} \cdots \int_0^{x_{i-1}} f(t) g^{(i)}(t)\, dt \cdots dx_2, \qquad i = 1, \ldots, l,$$

is i times ordinarily differentiable and $H^{(i)}(y) = f(y) g^{(i)}(y)$ for $i = 1, \ldots, l$. By Lemma 5, $fg^{(i)}$ is l times Peano differentiable at y. Therefore by Lemma 7, H is l times Peano differentiable at y and

$$H_l(y) = (H^{(i)})_{l-i}(y) = (f(y) g^{(i)}(y))_{(l-i)}.$$

Hence h is l times Peano differentiable at y and

$$h_l(y) = \sum_{j=0}^{l} (-1)^j \binom{l}{j} (fg^{(j)})_{(l-j)}(y),$$

and, by Lemma 6, $h_l(y) = f_l(y) g(y)$.

It remains to prove that $h_l(0)$ exists and that $h_l(0) = 0$. For $y \neq 0$

$$\begin{aligned} \frac{h(y)}{y^l} = \frac{1}{y^l} \Bigg\{ & \binom{l}{0} y^l \varepsilon_k(0, y) + (k-l) \binom{l}{1} \int_0^y t^{l-1} \varepsilon_k(0, t)\, dt \\ & + \cdots + (k-l)(k-l+1) \cdots (k-1) \\ & \cdot \binom{l}{l} \int_0^y \int_0^{x_2} \cdots \int_0^{x_{l-1}} \varepsilon_k(0, t) dt \cdots dx_2 \Bigg\}. \end{aligned}$$

Hence $\lim_{y \to 0}(h(y)/y^l) = 0$. Therefore $h(0) = h_1(0) = \cdots = h_l(0) = 0$. □

Now suppose that f has an lth Peano derivative in some neighborhood $\mathscr{O}$ of a point x and that $f_k(x)$ exists. Consider a function $T(y) = f(y) - f(x) - (y-x) f_1(x) - \cdots - (y-x)^k f_k(x)/k!$ and its translate $G(t) = T(x+t)$.

Then G satisfies the hypothesis of Lemma 8 and by that lemma the function H defined by

$$\begin{aligned} H(y) = & \binom{l}{0} G(y) g(y) - \binom{l}{1} \int_0^y G(t) g'(t)\, dt \\ & + \cdots + (-1)^l \binom{l}{l} \int_0^y \int_0^{x_2} \cdots \int_0^{x_{l-1}} G(t) g^{(l)}(t)\, dt \cdots dx_2 \quad \text{for } y \neq 0 \end{aligned}$$

and $H(0) = 0$ has an lth Peano derivative on $x - \mathscr{O}$.

Moreover, by the same lemma,

$$H_l(y) = \begin{cases} G_l(y)/y^{k-l} & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$$

But

$$G_l(t) = T_l(t+x) = f_l(t+x) - f_l(x) - t f_{l+1}(x) - \cdots - t^{k-l} \frac{f_k(x)}{(k-l)!}.$$

Therefore we have proved the following theorem.

Theorem 3. *Suppose that a function f in some neighborhood $\mathscr{O}$ of a point x has an lth Peano derivative in $\mathscr{O}$ and a kth Peano derivative at x, where*

$0 \le l \le k$. *Then the function* F *defined on* $\mathscr{O}$ *by*

$$F(y) = \begin{cases} (f_l(y) - \sum_{j=0}^{k-l}(y-x)^j(f_{l+j}(x))/j!)/(y-x)^{k-l} & \text{if } y \neq x, \\ 0 & \text{if } y = x \end{cases}$$

is an l*th Peano derivative.*

Corollary 4. *Suppose that a function* f *defined in some neighborhood* $\mathscr{O}$ *of a point* x *has a* $(k-1)$*th Peano derivative in* $\mathscr{O}$ *and* k*th Peano derivative at* x. *Then there exists a perfect set* $P \subset \mathscr{O}$ *of positive measure such that* x *is a bilateral point of accumulation of* P *and*

$$\lim_{y \in P, y \to x} \frac{f_{k-1}(y) - f_{k-1}(x)}{y - x} = f_k(x).$$

Proof. The function F from Theorem 3, applied with $l = k-1$ is a $(k-1)$th Peano derivative and hence Baire 1, Darboux, and has Denjoy property. Therefore, there is a perfect set P of positive measure such that x is a bilateral point of accumulation of P and such that F is continuous at x with respect to P. □

6. A_n CAN BE CHOSEN TO BE PERFECT

In this section we will prove that the sets A_n from the definition of $[\Delta']$ for Peano derivatives can be chosen to be perfect.

Let $y \in H(f, M, 1)$ be an isolated point of $H(f, M, 1)$. Then there is a $1 > \delta(y) > 0$ so that $(y - 2\delta(y), y + 2\delta(y)) \cap H(f, M, 1) = \{y\}$. Let P_y be a perfect set containing y so that y is a bilateral point of accumulation of P_y satisfying

$$\lim_{z \in P_y, z \to y} \frac{f_{k-1}(z) - f_{k-1}(y)}{z - y} = f_k(y)$$

and

$$\left| \frac{f_{k-1}(z) - f_{k-1}(y)}{z - y} - f_k(y) \right| \le 1 \quad \text{for every } z \in P_y.$$

Corollary 4 assures the existence of P_y. If $P_y \cap (y + \frac{1}{n+1}, y + \frac{1}{n}) \neq \varnothing$, for $n \in \mathbf{Z} \setminus \{-1, 0\}$, then by the Baire category theorem there is $Q_n(y) \subset P_y \cap (y + \frac{1}{n+1}, y + \frac{1}{n})$, such that Q_n is perfect and that there is $M_n \in \mathbf{N}$ with $Q_n(y) \subset H(f, M_n, 1)$. Let

$$Q_y = \bigcup_{n \in \mathbf{Z} \setminus \{-1, 0\}} Q_n(y) \cap (y - \delta^2(y), y + \delta^2(y)) \cup \{y\},$$

and let

$$H_M = H(f, M, 1) \cup \{Q_y : y \in H(f, M, 1), \ y \text{ is isolated in } H(f, M, 1)\}.$$

Theorem 4. H_M *is a perfect set, and* f_{k-1} *is differentiable on* H_M *relative to* H_M *with* $(f_{k-1}|_{H_M})'(x) = f_k(x)$.

Proof. By the construction of H_M we see that no point is an isolated point. Note that each of Q_y is perfect and that $Q_y \cap Q_z = \varnothing$ if $y, z \in H(f, M, 1)$ are two different isolated points of $H(f, M, 1)$. Suppose that H_M is not closed. Then there is a sequence $\{z_n\}$ and a point z such that $\lim_{n\to\infty} z_n = z$

and $\{z_n\} \cap H(f, M, 1) = \varnothing$, but then either there is a subsequence $\{z_{n_k}\} \subset \{z_n\}$ and $y \in H(f, M, 1)$ with y an isolated point of $H(f, M, 1)$ so that $\{z_{n_k}\} \subset Q_y$, or there is a sequence $\{y_{n_k}\} \subset H(f, M, 1)$ so that y_{n_k} is an isolated point of $H(f, M, 1)$ and $z_{n_k} \in Q_{y_{n_k}}$ for $k = 1, 2, \dots$. In the first case $z \in Q_y \subset H_M$, and in the second $y_{n_k} \to z$ and hence $z \in H(f, M, 1)$.

Now if $x \in H_M$ is an isolated point of $H(f, M, 1)$, then clearly f'_{k-1} at x relative to H_M exists and is equal to $f_k(x)$. If $x \in Q_y$ for some $y \in H(f, M, 1)$ where y is an isolated point of $H(f, M, 1)$, then there is $n \in \mathbf{Z}$ so that $x \in Q_n(y) \subset H(f, M_n(y), 1)$, and by the fact that there are two numbers $a < b$ so that $(a, b) \cap H_M = Q_n(y)$, we see that f'_{k-1} at x relative to H_M exists and is equal to $f_k(x)$.

Finally let $x \in H(f, M, 1)$, and x not an isolated point of $H(f, M, 1)$. Let $\varepsilon > 0$ be given. Then there is $\varepsilon > \eta > 0$ so that

$$\left|\frac{f_{k-1}(y) - f_{k-1}(x)}{y - x} - f_k(x)\right| < \varepsilon$$

whenever $y \in H(f, M, 1)$ and $|y - x| < \eta$.

Let y be an isolated point of $H(f, M, 1)$, and let $z \in Q_y$ with $|z - x| < \eta/2$. Since $|y - z| < \delta^2(y) < \delta(y)$ and $|y - x| > 2\delta(y)$, we have $\eta/2 > |x - z| \geq |x - y| - |y - z| > 2\delta(y) - \delta(y) = \delta(y)$. Hence $|y - x| \leq |y - z| + |z - x| < \delta(y) + \eta/2 < \eta$.

Now

$$\begin{aligned}
&\left|\frac{f_{k-1}(z) - f_{k-1}(x)}{z - x} - f_k(x)\right| \\
&\quad = \left|\left(\frac{f_{k-1}(y) - f_{k-1}(x)}{y - x} - f_k(x)\right)\frac{y - x}{z - x}\right. \\
&\qquad \left. + \left(\frac{f_{k-1}(z) - f_{k-1}(y)}{z - y} - f_k(y)\right)\frac{z - y}{z - x} + \frac{z - y}{z - x}(f_k(y) - f_k(x))\right| \\
&\quad \leq \left|\frac{f_{k-1}(y) - f_{k-1}(x)}{y - x} - f_k(x)\right|\left|1 - \frac{z - y}{z - x}\right| \\
&\qquad + \left|\frac{f_{k-1}(z) - f_{k-1}(y)}{z - y} - f_k(y)\right|\left|\frac{z - y}{z - x}\right| + \left|\frac{z - y}{z - x}\right|(|f_k(x)| + |f_k(y)|) \\
&\quad \leq \varepsilon\left(1 + \frac{\delta^2(y)}{\delta(y)}\right) + 1 \cdot \frac{\delta^2(y)}{\delta(y)} + \frac{\delta^2(y)}{\delta(y)}4M \\
&\quad \leq 2\varepsilon + \delta(y)(1 + 4M) \leq 2\varepsilon + \frac{\varepsilon}{2}(1 + 4M),
\end{aligned}$$

and since ε was arbitrary, we have that f'_{k-1} at x relative to H_M exists and equals $f_k(x)$. □

Acknowledgments

The author would to thank Professor C. E. Weil, his thesis advisor, for the generous help he offered during the preparation of this paper.

REFERENCES

1. S. Agronsky, R. Biskner, A. Bruckner, and J. Mařik, *Representations of functions by derivatives,* Trans. Amer. Math. Soc. **263** (1981), 493–500.
2. A. M. Bruckner, *Differentiation of real functions,* Lecture Notes in Math., vol. 659, Springer, Berlin and New York, 1978.
3. M. E. Corominas, *Contribution a la théorie de la dérivation d'order supérieur,* Bull. Soc. Math. France **81** (1953), 176–222.
4. H. Oliver, *The exact Peano derivative,* Trans. Amer. Math. Soc. **76** (1954), 444–456.
5. R. J. O'Malley, *Decomposition of approximate derivatives,* Proc. Amer. Math. Soc. **69** (1978), 243–247.
6. J. Mařik, *On generalized derivatives,* Real Anal. Exchange **3** (1977–78), 87–92.
7. ———, *Derivatives and closed sets,* Acta Math. Hungar. **43** (1984), 25–29.
8. S. Verblunsky, *On the Peano derivatives,* Proc. London Math. Soc. (3) **22** (1971), 313–324.
9. C. Weil, *On properties of derivatives,* Trans. Amer. Math. Soc. **114** (1965), 363–376.
10. ———, *On approximate and Peano derivatives,* Proc. Amer. Math. Soc. **20** (1969), 487–490.
11. ———, *A property for certain derivatives,* Indiana Univ. Math. J. **23** (1973/74), 527–536.
12. A. Zygmund, *Trigonometric series,* 2nd ed., Cambridge Univ. Press, Cambridge, 1959.

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