

ON THE SPECTRUM
OF THE NEUMANN LAPLACIAN OF LONG RANGE HORNS:
A NOTE ON THE DAVIES-SIMON THEOREM

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ABSTRACT. For a class of regions with cusps (e.g., $\Omega = \{(x, y) : x > 1, |y| < \exp(-x^\alpha)\}$, $0 < \alpha < 1$) we show that $\sigma_{ac}(-\Delta_N^\Omega) = [0, \infty)$ of uniform multiplicity one, $\sigma_{\text{sing}}(-\Delta_N^\Omega) = \emptyset$, and $\sigma_{pp}(-\Delta_N^\Omega)$ consists of a discrete set of embedded eigenvalues of finite multiplicity.

1. INTRODUCTION

This note is a contribution to the study of spectral properties of Neumann Laplacians on regions of the form

$$(1.1) \quad \Omega = \{(x, y) : x > 1, |y| < f(x)\},$$

where f is a strictly positive $C^2[1, \infty)$ function. The Neumann Laplacian on Ω , $-\Delta_N^\Omega = H$, is a unique selfadjoint operator in $L^2(\Omega)$ whose quadratic form is given by the closure of

$$(1.2) \quad q(\phi, \phi) = \int_{\Omega} |\nabla \phi|^2 dx,$$

on $C_0^2(\overline{\Omega})$. The spectral properties of H in regions (1.1) have been studied in [2, 5, 8] and, if specialized to the case when $f(x) = \exp(-x^\alpha)$, $\alpha > 0$, they can be stated as follows ($f \sim g$ stands for $\lim_{x \rightarrow \infty} f(x)/g(x) = 1$).

Theorem 1.1. (i) [5] H has a discrete spectrum iff $\alpha > 1$.

(ii) [8] If $N_E(H)$ denotes the number of eigenvalues of H which are less than E , then for $\alpha > 1$

$$(1.3) \quad N_E(H) \sim E^{1/2+1/(2(\alpha-1))} C(\alpha) + E \int_1^\infty \exp(-x^\alpha) dx,$$

where

$$C(\alpha) = \frac{1}{4(\alpha-1)\sqrt{\pi}} \left(\frac{\alpha}{2}\right)^{1/(1-\alpha)} \frac{\Gamma(1/(2(\alpha-1)))}{\Gamma(3/2 + 1/(2(\alpha-1)))}.$$

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(iii) [2] If $\alpha = 1$, $\sigma_{ac}(H) = [\frac{1}{4}, \infty)$ of uniform multiplicity one, $\sigma_{sing}(H) = \emptyset$, and $\sigma_{pp}(H)$ consists of a discrete set $0 = \lambda_0 < \lambda_1 \leq \dots \leq \lambda_n \rightarrow \infty$ of eigenvalues of finite multiplicity.

(iv) [2] If $0 < \alpha < \frac{1}{2}$, the results of part (iii) remain valid except that now $\sigma_{ac}(H) = [0, \infty)$.

In this paper we prove a theorem which, specialized to the above example, covers the range $\frac{1}{2} \leq \alpha < 1$; namely, we will show that all the conclusions of Theorem 1.1(iv) are valid for $0 < \alpha < 1$. Thus, at least in the above special case, we have a complete picture of what happens with the Neumann Laplacian (for some recent examples of surprising spectral properties of $-\Delta_N^\Omega$, even for bounded regions Ω , we refer to [6, 14]).

Our strategy (and our proof) follows closely the one of Davies and Simon [2]. The major observation made in [2] is that any spectral behavior which distinguishes H from the Dirichlet case (e.g., existence of a continuous spectrum) can come only from the subspace P consisting of functions which depend on the x variable only and which are in the form domain of H (such functions cannot be in the form domain of Dirichlet Laplacian). Restricted to such a subspace, the form (1.1) acts as

$$2 \int_1^\infty |u'(x)|^2 f(x) dx,$$

and viewed as a form on $L^2([1, \infty), 2f(x) dx)$ yields an operator which is unitarily equivalent to the one-dimensional Schrodinger operator on $L^2([1, \infty), dx)$ of the form

$$(1.4) \quad H_V = -\frac{d^2}{dx^2} + V(x), \quad V = \frac{1}{4} \left(\frac{f'}{f}\right) + \frac{1}{2} \left(\frac{f'}{f}\right)'$$

On the subspace orthogonal to P , H acts in effect as a Dirichlet Laplacian, and off-diagonal perturbation terms are controlled by the function

$$k(x) = |f'(x)| + f'(x)^2/f(x).$$

We say that a function g on $[1, \infty)$ is short-range if $g(x) = O(x^{-1-\varepsilon})$ for some $\varepsilon > 0$. Davies and Simon proved the following in [2]:

Theorem 1.2. (i) If $f \rightarrow 0$ and $k \rightarrow 0$ as $x \rightarrow \infty$, H has discrete spectrum if and only if H_V does.

(ii) If k and V are short range, we have $\sigma_{ac}(H) = [0, \infty)$ of uniform multiplicity, $\sigma_{sing}(H) = \emptyset$, and $\sigma_{pp}(H)$ consists of a discrete set $0 = \lambda_0 < \lambda_1 \leq \dots \leq \lambda_n \rightarrow \infty$ of embedded eigenvalues of finite multiplicity.

When $f(x) = \exp(-x^\alpha)$,

$$V(x) = \frac{\alpha^2}{4} x^{2(\alpha-1)} - \frac{\alpha(\alpha-1)}{2} x^{(\alpha-2)}$$

and Theorem 1.2 certainly yields parts (i), (iv) of Theorem 1.1. Case $\alpha = 1$ is somewhat special since then $V = \frac{1}{4}$: The argument of [2] still carries through, with the obvious shift of the essential spectrum. With some additional technical assumptions it was shown in [8] that in case (i) of Theorem 1.1 one has

$$(1.5) \quad N_E(H) \sim N(H_V) + \frac{E}{2} \text{Vol}(\Omega).$$

Using the semiclassical formula and (1.5) we get part (ii) of Theorem 1.1.

It is now easy to understand why $\alpha = \frac{1}{2}$ is the critical value: while for $0 < \alpha < \frac{1}{2}$ the potential V is short-range, for $\frac{1}{2} \leq \alpha < 1$ it becomes long-range (thus the name, long-range horns), and to treat perturbations one has to use the long-range scattering theory, which tends to be technically involved (particularly in the case when $V(x) \sim x^{-\beta}$, $0 < \beta < \frac{1}{2}$, which corresponds to the range $\frac{3}{4} \leq \alpha < 1$). The authors of [2] used the short-range Enss theory to treat perturbations and conjectured “that it is likely that one can modify ... [their] ... argument” to prove the analog of Theorem 1.2(ii) in the long-range case. By replacing the short-range Enss theory with the long-range one we extend Theorem 1.2 to the case where V is long-range and dilation analytic. The dilation analyticity assumption, although restrictive, covers the example of Theorem 1.1 and enables a particularly simple technical treatment, thanks to the work of Perry [10, 11].

In the sequel, H_0 stands for the one-dimensional free Laplacian. Let $u(\theta) : L^2(\mathbf{R}) \rightarrow L^2(\mathbf{R})$ be a unitary mapping defined as

$$u(\theta)\phi(x) = \exp(\theta/2)\phi(\exp(\theta)x).$$

A potential \widehat{V} , defined on the whole real line, is called dilation analytic [11, 12] if \widehat{V} is H_0 -compact and the operator-valued function

$$C(\theta) = u(\theta)\widehat{V}u(\theta)^{-1}(H_0 + 1)^{-1}$$

extends to an analytic operator-valued function on the strip $\mathcal{S}(a) = \{z : -a < \text{Im}(z) < a\}$ for some $a > 0$.

With V as in (1.4), we define

$$\widehat{V}(x) = V(|x| + 1), \quad x \in \mathbf{R}.$$

In the sequel we assume that f is $C^3[1, \infty)$ so that V is a differentiable function. Our main result is

Theorem 1.3. *Suppose that k and V' are short range and that \widehat{V} is dilation analytic. Then all conclusions of Theorem 1.1(iv) remain valid.*

Since, for $f(x) = \exp(-x^\alpha)$,

$$\widehat{V}(x) = \frac{\alpha^2}{4}(|x| + 1)^{2(\alpha-1)} - \frac{\alpha(\alpha-1)}{2}(|x| + 1)^{\alpha-2}$$

is certainly dilation analytic [11] if $0 < \alpha < 1$, the above theorem applies to the range of α not covered by Theorem 1.1.

We finally remark that all the above-mentioned results have natural extensions to higher-dimensional nonsymmetric regions, as well as to manifolds with metric cusps at infinity [2, 6, 8].

2. PROOF OF THEOREM 1.3

Let H_V be the operator (1.4) with the Neumann boundary condition at 1 and denote by

$$D = \{\psi : \psi \in C^2[1, \infty), \psi'(1) = 0\}$$

its domain of essential selfadjointness. Let $J : L^2(\mathbf{R}) \rightarrow L^2(\Omega)$ be the embedding $(J\phi)(x) = \phi(x)/\sqrt{2f(x)}$, and denote $Q = 1 - JJ^*$.

The first ingredient in our argument is the following result of Davies and Simon [2].

Theorem 2.1. (i) If $\phi \in D$,

$$\|(H+1)^{-1/2}(HJ - JH_V)\phi\| \leq 2 \cdot \|f'\phi\|_2 + \|((f')^2/f)\phi\|_2 + C|\phi(1)|,$$

where C is a positive constant.

(ii) $Q(H+1)^{-1/2}$ is a compact operator in $L^2(\Omega)$.

(iii) If g is a continuous function on $R \cup \{\infty\}$, $g(H)J - Jg(H_V)$ is a compact operator in $L^2(\Omega)$.

We now use the decay of V' and the dilation analyticity of \widehat{V} to obtain a useful decomposition of V into a short-range part and a smooth long-range dilation analytic part [10]. \widehat{V} can be decomposed as

$$(2.1) \quad \widehat{V}(x) = \widehat{V}_L(x) + \widehat{V}_S(x),$$

where $\widehat{V}_L(x)$ is a C^∞ dilation analytic potential with all derivatives bounded,

$$\widehat{V}'(x) = O(|x|^{-1-\varepsilon}),$$

and $\widehat{V}_S(x)$ is a short-range function. Both parts \widehat{V}_L , \widehat{V}_S can be chosen to be even functions around 0, and thus they induce the decomposition

$$V(x) = V_L(x) + V_S(x),$$

with obvious inheritance of properties. We refer to [10] for a proof of (2.1). \widehat{V}_L is defined as the Weierstrass transform of \widehat{V} ,

$$\widehat{V}_L(x) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-y)^2}{4}\right) \widehat{V}(y) dy.$$

Let

$$\widehat{H}_L = H_0 + \widehat{V}_L, \quad H_L = -\frac{d^2}{dx^2} + V_L,$$

where the second operator acts on $L^2([1, \infty), dx)$ with the Neumann boundary condition at 1. The third ingredient in the argument consists of Perry's propagation estimates [10, 11] on $\exp(-it\widehat{H}_L)$ (see also [1]), which automatically yield the propagation estimates on $\exp(-itH_L)$. Let A be a scale transformation around $x = 1$ [1, 10],

$$A = \frac{1}{2}((x-1) \cdot p + p \cdot (x-1)),$$

where $p = -iD = -i\partial/\partial x$ in the x -space representation. A is essentially selfadjoint on $C_0^\infty(R)$, $\sigma(A) = \sigma_{ac}(A) = (-\infty, \infty)$, and A leaves invariant the subspace of functions which are even around 1. By P_\pm we denote the spectral projections of A on $\pm(0, \infty)$ and by P_\pm^a the spectral projections on $\pm(a, \infty)$. Obviously, $P_+ + P_- = 1$. It is a standard result [12] that operators \widehat{H}_L , H_L have no (strictly) positive eigenvalues, that $\sigma_{sing}(\widehat{H}_L) = \sigma_{sing}(H_L) = \emptyset$, and that $\sigma_{ac}(\widehat{H}_L) = \sigma_{ac}(H_L) = [0, \infty)$. Since \widehat{V} is even around 0, \widehat{H}_L preserves the subspace of functions which are in its domain and are even around 0. After translation by 1, the restriction of \widehat{H}_L to that subspace coincides with H_L . In

the sequel we will use Enss' notation where $F(M)$ stands for the characteristic function of the set M . P_{ac} will denote the spectral projection on the absolutely continuous subspace of H_L . Viewing H_L as a restriction of \hat{H}_L (keeping the above translation in mind) we have

Theorem 2.2. (i) *Let g be a C^∞ function with support in $[\alpha, \beta]$, $\alpha > 0$. Then for any $\delta > 0$, $N > 0$, and $a \in \mathbf{R}$,*

$$(2.2) \quad \|F(1 \leq x < |t|^{1-\delta}) \exp(itH_L)g(H_L)P_{\pm}^a\| = O(|t|^{-N})$$

as $t \rightarrow \pm\infty$.

$$(ii) \text{ s-}\lim_{t \rightarrow \pm\infty} P_{\pm}^a \exp(-itH_L) = 0.$$

The above theorem is a result by Perry [10, 11].

Since H_L plays the role of the "free Laplacian" in the discussion below, it is worthwhile to rephrase part (i) of Theorem 2.1 as

$$\|(H + 1)^{-1/2}(JH - JH_L)\phi\| \leq 2 \cdot \|f'\phi\|_2 + \|((f')^2/f)\phi\|_2 + \|V_L\phi\|_2 + C|\phi(1)|$$

and to note, concerning part (iii), that $g(H)J - Jg(H_L)$ is a compact operator in $L^2(\Omega)$ (since $g(H_V) - g(H_L)$ is compact).

Since V_L is bounded from below, without loss of generality we can assume $V_L \geq 0$. The final technical lemma is the following

Lemma 2.3.

$$\|(H + 1)^{-1/2}(HJ - JH_L)(H_L + 1)^{-1}F(x > R)\| = O(R^{-1-\epsilon}).$$

The proof is standard [1, Lemma 5.4], using Theorem 2.1(i) and the fact that k, V_S are short-range.

From now on, the proof follows line-by-line the argument of Davies and Simon [2]. We give the details in the appendix for the reader's convenience.

APPENDIX

Step 1. $\text{s-}\lim_{t \rightarrow \pm\infty} (H + 1)^{-1/2} \exp(itH)J \exp(-itH_L)P_{ac}$ exists. Let us consider only the limit $t \rightarrow \infty$; a similar consideration applies to the other one. By Cook's criterion, it suffices to show that

$$\int_0^\infty \|(H + 1)^{-1/2}(HJ - JH_L) \exp(-itH_L)g(H_L)P_{+}^a\| dt < \infty,$$

where g is as in Theorem 2.2, since $\bigcup_{a,g} \text{Ran } g(H_L)P_{+}^a$ is dense in $\mathcal{H}_{ac}(H_L)$. Denote

$$A(t) = \|(H + 1)^{-1/2}(HJ - JH_L) \exp(-itH_L)g(H_L)P_{+}^a\|.$$

Choosing $\delta < \epsilon$ in (2.2), we estimate, using Theorem 2.2 and Lemma 2.3,

$$\begin{aligned} A(t) &\leq \|(H + 1)^{-1/2}(HJ - JH_L)(H_L + 1)^{-1}F(x \geq t^{-1-\delta})\| \\ &\quad + \|F(1 \leq x < t^{-1-\delta}) \exp(-itH_L)(H_L + 1)g(H_L)P_{+}^a\| \\ &\leq O(t^{-1-\epsilon+\delta}). \end{aligned}$$

Step 2. The wave operators $\Omega^\pm = \text{s-}\lim_{t \rightarrow \mp\infty} \exp(itH)J \exp(-itH_L)P_{ac}$ exist. $(H + 1)^{-1/2}J - J(H_L + 1)^{-1/2}$ is compact by Theorem 2.1, since

$$\begin{aligned} &(H + 1)^{-1/2}J - J(H_L + 1)^{-1/2} \\ &= (H + 1)^{-1/2}J - J(H_V + 1)^{-1/2} + J(H_V + 1)^{-1/2} - J(H_L + 1)^{-1/2}. \end{aligned}$$

Thus

$$s\text{-}\lim_{t \rightarrow \mp\infty} \exp(-itH)((H+1)^{-1/2}J - J(H_L+1)^{-1/2})\exp(-itH_L)P_{ac} = 0.$$

By Step 1, $\Omega^\pm\phi$ exist if $\phi \in \text{Ran}(H_L+1)^{-1/2}P_{ac}$, and the last set coincides with $\mathcal{H}_{ac}(H_L)$.

Step 3. $(H+1)^{-1/2}(\Omega^\pm - J)g(H_L)P_\pm$ is a compact operator.

First, since $(H_L+1)^{-1/2}(\exp(itH_L)J\exp(-itH_L) - J)g(H_L)P_\pm$ converges in the operator norm as $t \rightarrow \pm\infty$, it suffices to show that each of these operators is compact. On the other hand, they are integrals of operators of the form

$$(A.1) \quad (H+1)^{-1/2}(HJ - JH_L)\exp(-itH_L)g(H_L)P_\pm,$$

and thus it suffices to show that the operators (A.1) are compact. But that is a consequence of part (iii) of Theorem 2.1 and of the fact that V_S is short-range.

Step 4. If $\phi_n \in \mathcal{H}_{ac}(H)^\perp$, with $\|(H+1)\phi_n\|$ bounded and $(H+1)^{1/2}\phi_n \rightarrow 0$ weakly, then $\|\phi_n\| \rightarrow 0$ as $n \rightarrow \infty$.

First, $(\Omega^\pm)^*\phi_n = 0$ for all n , and thus for any $C_0^\infty((0, \infty))$ function g , Step 3 yields

$$P_\pm g(H_L)J^*\phi_n \rightarrow 0.$$

Since $P_+ + P_- = 1$ and $g(H_L)J^* - J^*g(H)$ is a compact operator, we obtain that $J^*g(H)\phi_n \rightarrow 0$. Since $\|(H+1)\phi_n\|$ is bounded, we estimate

$$(A.2) \quad \|g(H)\phi_n - \phi_n\| \leq C \cdot \|(g(H) - 1) \cdot (H+1)^{-1}\|.$$

Since g is arbitrary, the left-hand side of (A.2) can be made arbitrarily small, and thus $J^*\phi_n \rightarrow 0$, and so $JJ^*\phi_n \rightarrow 0$. By Theorem 2.1, $Q\phi_n = (1 - JJ^*)\phi_n \rightarrow 0$, and we conclude that $\phi_n \rightarrow 0$.

Step 5. $\sigma_{\text{sing}}(H) = \emptyset$ and in any finite interval H has only finitely many eigenvalues. If any of the statements is not valid, we can construct an orthonormal sequence ϕ_n so that $\phi_n \in \mathcal{H}_{ac}(H)^\perp$, $\|(H+1)\phi_n\|$ is bounded, and $(H+1)^{1/2}\phi_n \rightarrow 0$ weakly. Step 4 implies then that $\phi_n \rightarrow 0$, which contradicts the fact that the sequence ϕ_n is orthonormal.

Step 6. $\sigma_{ac}(H)$ has multiplicity one. It suffices to show that $\text{Ran}\Omega^+ = \mathcal{H}_{ac}(H)$. Suppose that it is not, namely, that we can find a nonzero vector $\phi \in \mathcal{H}_{ac}(H) \cap (\text{Ran}\Omega^+)^\perp \cap D(H)$. Define $\phi_n = \exp(-inH)\phi$. Part (ii) of Theorem 2.2 yields

$$P_\pm(\Omega^\pm)^*\phi_n = P_\pm \exp(-inH_L)P_{ac}(\Omega^\pm)^*\phi \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Thus, as in Step 4, $\|\phi_n\| \rightarrow 0$, and we derive that $\phi = 0$, a contradiction.

Step 7. There exists a discrete set of embedded eigenvalues. This is a consequence of symmetry of Ω with respect to the axes $y = 0$. Let E be the subspace of $L^2(\Omega)$ consisting of those functions which are even under reflection $(x, y) \rightarrow (x, -y)$. That subspace is invariant under Q and H , and thus H restricted to it has a compact resolvent by Theorem 2.1.

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