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A FIXED POINT PROPERTY OF l_1-PRODUCT SPACES

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ABSTRACT. Let X_1 and X_2 be Banach spaces, and let $X_1 \times X_2$ be equipped with the l_1-norm. If the first space X_1 is uniformly convex in every direction, then $X_1 \times X_2$ has the fixed point property for nonexpansive mappings (FPP) if and only if $\mathbb{R} \times X_2$ (with the l_1-norm) does. If X_1 is merely strictly convex, $(\mathbb{R} \times X_2)$ has the FPP, and $C_i \subset X_i$ are weakly compact and convex with the FPP (for $i = 1, 2$), then the fixed point set of every nonexpansive mapping $T: C_1 \times C_2 \to C_1 \times C_2$ is a nonexpansive retract of $C_1 \times C_2$.

INTRODUCTION

Our purpose in this paper is to provide sufficient conditions for a product of two Banach spaces $X_1 \times X_2$ equipped with the l_1-norm to have the fixed point property for nonexpansive mappings (FPP) and for the fixed point sets of such mappings to be nonexpansive retracts.

First we show that if the first space X_1 is uniformly convex in every direction, then $X_1 \times X_2$ has the FPP if and only if $\mathbb{R} \times X_2$ (with the l_1-norm) does. Next we prove that if X_1 is merely strictly convex, $\mathbb{R} \times X_2$ has the FPP, and $C_i \subset X_i$ are weakly compact and convex with the FPP, then the fixed point set of every nonexpansive mapping $T: C_1 \times C_2 \to C_1 \times C_2$ is a nonexpansive retract of $C_1 \times C_2$. The same conclusion holds if both spaces are strictly convex. It turns out that there are many spaces X_2 such that $\mathbb{R} \times X_2$ does indeed have the FPP. We include a list of spaces with this property. Finally, we show that if X_1 has the Schur property and X_2 has the semi-Opial property, then $(X_1 \times X_2)_1$ again has the FPP.

For a study of the FPP in product spaces endowed with norms generated by a strictly convex norm on the positive cone of $\mathbb{R}^2$, see [26], which also contains a comprehensive bibliography for this kind of problem. The case of the l_∞-norm is still only partially understood [13, 23, 27, 28, 36].

EXISTENCE OF FIXED POINTS

Recall that if C is a nonempty subset of a Banach space $(X, \|\cdot\|)$, then $T: C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all x and y

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in C. We say that a nonempty weakly compact convex subset C of a Banach space $(X, \|\cdot\|)$ has the fixed point property for nonexpansive mappings (FPP) if every nonexpansive self-mapping T of C has a fixed point. A Banach space $(X, \|\cdot\|)$ is said to have the FPP if every nonempty weakly compact convex subset of X has the FPP.

The following lemma (due to Goebel and Karlovitz) will be very useful in the sequel.

Lemma [12–14, 18]. *Assume that C is a nonempty, weakly compact, and convex subset of the Banach space $(X, \|\cdot\|)$. If $T: C \to C$ is nonexpansive and C is minimal, then, for every sequence $\{x_n\}$ in C with $\lim_n \|x_n - Tx_n\| = 0$, $\lim_n \|x - x_n\| = \operatorname{diam} C$ for each $x \in C$.*

Here we recall that if C is a nonempty weakly compact convex subset of the Banach space $(X, \|\cdot\|)$ and $T: C \to C$ is nonexpansive, then we say that C is a minimal set with respect to T (or simply minimal) if it contains no proper closed convex subsets which are invariant under T.

We need one more definition. For a Banach space $(X, \|\cdot\|)$ and a fixed element $z \in X$ with $\|z\| = 1$, let the modulus of convexity of X in the direction $z \in X$ be the function $\delta_z: [0, 2] \to [0, 1]$ defined by

$$\delta_z(\varepsilon) = \inf\{1 - \tfrac{1}{2}\|x+y\| : \|x\| \le 1,\ \|y\| \le 1,\ \|x-y\| \ge \varepsilon \text{ and } x - y = tz \text{ for some } t \in \mathbb{R}\}.$$

If $\delta_z(\varepsilon) > 0$ for all $\varepsilon > 0$ and all such z, then X is called uniformly convex in every direction [9, 10] (see also [1, 6, 7, 29]).

Theorem 1. *Let $(X_1, \|\cdot\|_1)$ be a Banach space which is uniformly convex in every direction. If $(X_2, \|\cdot\|_2)$ is a Banach space such that $\mathbb{R} \times X_2$ with the l_1-norm has the FPP, then $X_1 \times X_2$ with the l_1-norm also has the FPP.*

Proof. Let C be a nonempty, weakly compact, and convex subset of $X_1 \times X_2$, and let $T: C \to C$ be a nonexpansive mapping. We can assume that C is minimal, $\operatorname{diam} C > 0$, and $0 \in C$. Let $\{x_n\} = \{(x_{n1}, x_{n2})\}$ be an approximate fixed point sequence in C (i.e., $\lim_n \|x_n - Tx_n\| = 0$). By the separability of C and the Goebel-Karlovitz Lemma we can also assume that for every $x = (x_1, x_2) \in C$ the following limits exist:

$$\lim_n \|x - x_n\| = \operatorname{diam} C, \quad \lim_n \|x_1 - x_{n1}\|_1, \quad \text{and} \quad \lim_n \|x_2 - x_{n2}\|_2.$$

Now we show that $\operatorname{Proj}_{X_1} C$ is a segment. To this end, assume that

$$y_{11} = \operatorname{Proj}_{X_1}(y_{11}, y_{12}) = \operatorname{Proj}_{X_1} y_1 \quad \text{and} \quad y_{21} = \operatorname{Proj}_{X_1}(y_{21}, y_{22}) = \operatorname{Proj}_{X_1} y_2$$

are two points in $\operatorname{Proj}_{X_1} C$ such that 0, y_{11}, and y_{21} are not collinear. Then without loss of generality we may assume that

$$\lim_n \|y_{11} - x_{n1}\|_1 = \lim_n \|y_{21} - x_{n1}\|_1 > 0,$$

and therefore we obtain the following contradiction:

$$\begin{aligned}\operatorname{diam} C &= \lim_n \|\tfrac{1}{2}(y_1 + y_2) - x_n\| \\ &= \lim_n [\|\tfrac{1}{2}(y_{11} + y_{21}) - x_{n1}\|_1 + \|\tfrac{1}{2}(y_{12} + y_{22}) - x_{n2}\|_2] \\ &< \tfrac{1}{2} \lim_n [\|y_{11} - x_{n1}\|_1 + \|y_{12} - x_{n2}\|_2] \\ &\quad + \tfrac{1}{2} \lim_n [\|y_{21} - x_{n1}\|_1 + \|y_{22} - x_{n2}\|_2] \\ &= \tfrac{1}{2} \lim_n \|y_1 - x_n\| + \tfrac{1}{2} \lim_n \|y_2 - x_n\| = \operatorname{diam} C.\end{aligned}$$

Hence $C \subset \mathbb{R} \times X_2$, and therefore it consists of one point. The proof is complete.

Nonexpansive retracts

Let $(X, \|\cdot\|)$ be a Banach space and C be a nonempty subset of X. We say that a nonempty subset D of C is a nonexpansive retract of C if there exists a nonexpansive mapping $r\colon C \to D$ such that $r|_D$ is the identity mapping.

Theorem 2. *Let $(X_1, \|\cdot\|_1)$ be a strictly convex Banach space, and let $(X_2, \|\cdot\|_2)$ be a Banach space such that $\mathbb{R} \times X_2$ with the l_1-norm has the FPP. Let the product space $X_1 \times X_2$ be furnished with the l_1-norm. If $\varnothing \neq C_1 \subset X_1$ and $\varnothing \neq C_2 \subset X_2$ are weakly compact and convex and C_1 has the FPP, then the fixed point set of every nonexpansive self-mapping T of $C_1 \times C_2$ is a nonexpansive retract of $C_1 \times C_2$.*

Proof. We apply Bruck's method [3]. It is sufficient to prove that

$$\forall x_0 \in C_1 \times C_2 \exists F \in N(\operatorname{Fix} T)\ F x_0 \in \operatorname{Fix} T = \{x \in C_1 \times C_2 \colon Tx = x\},$$

where

$$N(\operatorname{Fix} T) = \{F\colon C_1 \times C_2 \to C_1 \times C_2 \colon F \text{ is nonexpansive and } \operatorname{Fix} T \subset \operatorname{Fix} F\}.$$

(In [24] Kirk and Martinez Yanez proved that $\operatorname{Fix} T \neq \varnothing$; see also [21].) In $N(\operatorname{Fix} T)$ with the partial order given by

$$\begin{aligned} F \leq G = \{&\text{either } F = G \text{ or } \|Fx - Fy\| \leq \|Gx - Gy\| \text{ for} \\ &\text{all } x, y \in C_1 \times C_2 \text{ with strict inequality holding} \\ &\text{for at least one pair of points } x, y \in C_1 \times C_2\},\end{aligned}$$

there exists a minimal element $r = (r_1, r_2)$. Fix this r and consider the subfamily

$$N'(\operatorname{Fix} T) = \{r' \in N(\operatorname{Fix} T) \colon \forall x, y \in C_1 \times C_2\ \|r'x - r'y\| = \|rx - ry\|\}$$

of $N(\operatorname{Fix} T)$. The subfamily $N'(\operatorname{Fix} T)$ is nonempty, convex, and compact in the topology of weak pointwise convergence. In addition, we have $Tr' \in N(\operatorname{Fix} T)$ for every $r' \in N'(\operatorname{Fix} T)$. Now choose and fix $x_0 \in C_1 \times C_2$. Without loss of generality we can assume that $0 \in \operatorname{Fix} T$ and $r_1(x_0) \neq 0$. Set $D = \{r'(x_0) \colon r' \in N'(\operatorname{Fix} T)\}$. D is weakly compact, convex, and T-invariant. To complete the proof we claim that $D \subset \mathbb{R} \times X_2$. Indeed, for $r' = (r_1', r_2') \in N(\operatorname{Fix} T)$ we get

$$\begin{aligned}\|r_1 x_0\|_1 + \|r_2 x_0\|_2 &= \|r x_0\| = \|r x_0 - r0\| = \|r' x_0 - r'0\| \\ &= \|r' x_0\| = \|r_1' x_0\|_1 + \|r_2' x_0\|_2.\end{aligned}$$

Our claim now follows by the minimality of r and the strict convexity of X_1.

In a similar way we can prove another result in this direction.

Theorem 3. *If* $(X_1, \|\cdot\|_1)$ *and* $(X_2, \|\cdot\|_2)$ *are strictly convex Banach spaces, and* $\varnothing \neq C_1 \subset X_1$ *and* $\varnothing \neq C_2 \subset X_2$ *are weakly compact, convex, and with the FPP, then* $\operatorname{Fix} T$ *is a nonexpansive retract of* $C_1 \times C_2$ *for every nonexpansive* $T: C_1 \times C_2 \to C_1 \times C_2$.

Proof. In this case the set D defined in the proof of Theorem 2 is a subset of $\mathbb{R} \times \mathbb{R}$.

Examples of $\mathbb{R} \times X$ with the FPP

First we give a few known results. In [30, 31] Landes proved that $\mathbb{R} \times X$ with the l_1-norm has weakly normal structure and, therefore, the FPP if X has one of the following properties:

(i) X has uniform normal structure [11].
(ii) X has a coefficient of convexity $\varepsilon_0 < 1$ [5, 13].
(iii) X is uniformly convex in every direction [9, 10].
(iv) X is k-uniformly rotund [37].
(v) X is nearly uniformly convex [13, 17].
(vi) $\mathrm{BS}(X) > 1$, where $\mathrm{BS}(X)$ is the bounded sequence coefficient of X [5].
(vii) $\mathrm{WCS}(X) > 1$, where $\mathrm{WCS}(X)$ is the weakly convergent sequence coefficient of X [5].
(viii) X has a basis with the Gossez-Lami Dozo property [4, 13, 15].
(ix) X satisfies Bynum's condition [4].
(x) X has Opial's property [14, 16, 34, 35].

Next it is easy to observe that Banach spaces with the following properties can also play the role of X in the l_1-product of $\mathbb{R} \times X$ with the FPP:

(a) X has the Schur property [8].

(b) X has an unconditional basis $\{e_i\}$ with an unconditional constant $\lambda < \frac{1}{2}(\sqrt{33} - 3)$ [32], where

$$\lambda = \sup\left\{\left\|\sum_{i=1}^{\infty} \varepsilon_i \xi_i e_i\right\| : \left\|\sum_{i=1}^{\infty} \xi_i e_i\right\| = 1,\ \varepsilon_i = \pm 1\right\}.$$

(c) X is superreflexive with an unconditional basis $\{e_i\}$ satisfying $c = 1$ [32], where

$$c = \sup\{\|P_F\| : F \subset \mathbb{N}\}$$

and

$$P_F x = \sum_{i \in F} \xi_i e_i \quad \text{for } x = \sum_{i=1}^{\infty} \xi_i e_i \in X.$$

(d) X is a Banach space having an unconditional basis $\{e_i\}$ satisfying $c(\lambda + 2) < 4$ [19].

(e) X is a Banach space that has an unconditional basis $\{e_i\}$ which is unconditionally monotone, and X has the alternate Banach-Saks property [1, 20].

(f) X has a Schauder finite-dimensional decomposition with $\beta_p(X) < 2^{1/p}$ for some $p \in [1, +\infty)$ [1, 22], where

$$\beta_p(X) = \inf\{\lambda : (\|x\|^p + \|y\|^p)^{1/p} \le \lambda\|x + y\| \text{ for every } x \text{ and } y \in X \text{ such that } \operatorname{supp}(x) < \operatorname{supp}(y)\}.$$

We conclude the present paper by introducing a new class of Banach spaces. As we will prove, the l_1-product of a space in this class with $\mathbb{R}$ has the FPP.

Definition. Let $(X, \|\cdot\|)$ be a Banach space. We say that $(X, \|\cdot\|)$ has the semi-Opial property if for any bounded nonconstant sequence $\{x_n\}$ with $\lim_n \|x_n - x_{n+1}\| = 0$ there exists a subsequence $\{x_{n_i}\}$ such that $\text{w-}\lim_i x_{n_i} = x$ and $\lim_i \|x - x_{n_i}\| < \operatorname{diam}\{x_n\}$.

The following spaces have the semi-Opial property:

(1) X has Opial's property.
(2) X has uniformly normal structure [33].
(3) X is nearly uniformly convex [17].
(4) $X = X_\beta$, where $1 < \beta < 2$, $X_\beta = (l^2, \|\ \|_\beta)$, and, for $x \in l^2$, $\|x\|_\beta = \max(|x|_2, \beta|x|_\infty)$ [2, 13, 26].
(5) X is the James quasi-reflexive space [39].

Theorem 4. *Let $(X_1, \|\cdot\|_1)$ be a Banach space with the Schur property, and let $(X_2, \|\cdot\|_2)$ be a Banach space with the semi-Opial property. If $X_1 \times X_2$ is the product space endowed with the l_1-norm, then it has the FPP.*

Proof. As usual, we assume that $T: C \to C$ is nonexpansive on a nonempty, weakly compact, convex subset C of $X_1 \times X_2$ and that C is minimal for T. Assume that $\operatorname{diam} C > 0$. Then there exists an approximate fixed point sequence $\{x_n\} = \{(x_{n_1}, x_{n2})\}$ such that $\lim_n \|x_n - x_{n+1}\| = 0$. It is obvious that in this case $\operatorname{diam}\{x_{n2}\} > 0$. By the semi-Opial property of X_2 and the Schur property of X_1, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\text{w-}\lim_i x_{n_i} = x = (x_1, x_2)$$

and

$$\lim_i \|x_{n_i,2} - x_2\|_2 < \operatorname{diam}\{x_{n,2}\}.$$

By the Goebel-Karlovitz Lemma this yields the following contradiction:

$$\begin{aligned}\operatorname{diam} C = \lim_i \|x_{n_i} - x\| &= \lim_i \|x_{n_i,2} - x_2\|_2 \\ &< \operatorname{diam}\{x_{n,2}\} \le \operatorname{diam}\{x_n\} = \operatorname{diam} C.\end{aligned}$$

The diameter of C must therefore be equal to 0, and the proof is complete.

Remark. In Theorem 4 the l_1-norm can be replaced by a norm in $X_1 \times X_2$ satisfying the following conditions:

(I) The restrictions of the norm on $X_1 \times X_2$ to X_1 and X_2 are the initial norms of X_1 and X_2.
(II) The natural projection on X_2 has norm 1.

Moreover, if X_1 is finite dimensional, then (I) can be replaced by

(I′) The restriction of the norm on $X_1 \times X_2$ to X_2 is the initial norm on X_2.

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